Integrability of Bukhvostov-Lipatov model and ODE/IQFT correspondence

Boris A. Runov

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I declare that the work contained in this thesis is my own original research, produced in collaboration with my supervisor Prof. Vladimir Bazhanov and Prof. Sergei Lukyanov between February 2015 and June 2018 at the Australian National University. All materials taken from other sources are referenced and aknowledged as such. I also declare that none of the material contained in this thesis has been submitted for another degree at this or another university.

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Aknowlegments

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Abstract

We consider Bukhvostov-Lipatov model, an integrable quantum field theory in two dimensions that arises as an approximation to O(3) NLSM. We compute its vacuum energy on a cylinder with twisted boundary conditions in weak coupling limit using renormalized perturbation theory, and in the short distance limit using conformal perturbation theory. The exact solution of this model via coordinate Bethe ansatz is provided. Two different regularizations of Bethe ansatz equations are constructed. The vacuum state is constructed and the vacuum energy is computed within both regularizations using numerical methods. Bethe ansatz equations governing the vacuum state are shown to coincide with functional relations between connection coefficients of auxiliary linear problem for an integrable classical PDE known as modified sinh-Gordon equation. Based on this correspondence the system of nonlinear integral equations equivalent to the full system of Bethe ansatz equations is derived for massive and conformal cases. This system of NLIE was solved numerically. We have also used it to investigate analytically the properties of solution describing vacuum state. Finally, we have derived a formula that expresses the vacuum energy of Bukhvostov-Lipatov model in terms of the regularized area of constant mean curvature surface embedded into AdS_3 space.

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Chapter 1

Introduction

The most advanced experimentally confirmed model describing electromagnetic, weak and strong interactions is the Standard Model of elementary particles [1–13]. The part of the model describing strong interactions is known as quantum chromodynamics [14, 15]. It is a gauge theory with gauge group SU(3) [16] and the matter in fundamental representation of gauge group. The gauge bosons in this theory are called gluons and fermions are known as quarks. High-energy behaviour of quantum chromodynamics was extensively studied both theoretically and experimentally, with extremely good agreement between theory and experiment [17]. However, despite perceived simplicity of the Lagrangian of the QCD, analytical treatment of low-energy behaviour from the first principles is still impossible in many cases, and we have to rely on lattice simulations and various phenomenological effective theories (see p.e. [18, 19]) for description of nuclea and hadrons. The phenomenon of "quark confinement", i.e. the fact that individual quarks can not be directly observed on experiment the same way as mesons and barions, is not yet properly explained [20, 21].

The main obstacle is the behaviour of the coupling constant of the QCD under the renormalization group flow [22–31]. In this approach to the renormalization a physical quantum field theory is always defined at some finite momentum scale. The change of the scale yields (for a renormalizable theory) a theory described by similar Lagrangian differing only in values of finite number of parameters such as masses and coupling constants. For the quantum chromodynamics the physical coupling constant tends to zero in short distance limit (such behaviour was labeled "asymptotic freedom") [32–35], and grows significantly at large scales. Perturbatively defined renormalization group equation implies that coupling constant diverges at some finite scale known as Λ_{QCD} [36], which serves as the characteristic scale where infrared effects become dominating. As a consequence, perturbation theory, which is the primary method to extract numerical predictions for physical observables in the theory, is effective at large energies but completely useless at energies below Λ_{QCD}. So new mathematical methods that do not rely directly on pertrbative expansion in coupling constant are required.

One direction of research is based on the idea introduced in [37, 38]: to describe the low-energy behaviour of the strongly coupled Yang-Mills theory in terms of special solutions to classical euclidean equations of motion governing the field theory, known as instantons. The instantons are the solutions that interpolate between two classical configurations of the field with lowest energy (sometimes referred two as "classical vacua"). Often they are of topological nature. At small energies they are shown to give significant contribution to path integral (in comparison with perturbative contributions, which are given by trajectories returning to the same classical vacuum via topologically trivial path in configuration space). This is related to the fact that perturbation theory series encountered in a quantum field theory are typically asymptotic, which means that they not converge. The exact answer can be written as a sum of perturbative series and other series of non-perturbative nature. These objects are sometimes called "trans-series" and the concept of that summation is commonly referred to as "resurgence" [39].

This approach allowed to derive quantitative description of confinement in some toy models, but was not sufficient to deal with QCD itself, because the task of summing instanton contributions is extremely complicated in this case. To the day, it could be done explicitly only for abelian [37] or supersymmetric [40] theories. The natural playground for this task is provided by $O(N)$ and $\mathbb{C}P^{N-1}$ nonlinear sigma models in two dimensions, which exhibit strong similarities with Yang-Mills theory with gauge group $SU(N)$ [41,42]. In particular, the mass gap in $O(N)$ sigma models was computed explicitly in [43–46].

The starting point of this study is $O(3)$ non-linear sigma model in two space-time dimensions. For this theory, the contribution of instanton configurations to partition function was computed (including quadratic fluctuations) by Fateev et al [47]. It was shown to be equal to partition function of free massive Dirac fermions in two dimensions. Later this result was extended by Bukhvostov and Lipatov [48], who considered configurations involving both instantons and anti-instantons (more precisely, the so-called weakly interacting instanton–anti-instanton configurations). Such configurations are not solutions to euclidean equations of motion, but are close enough to them to give substantial contribution to the partition function. Similarly to the result of [47], the answer could be expressed as partition function of a two-dimensional fermionic QFT. This theory, which we call here the Bukhvostov-Lipatov model, turned out to be exactly solvable [48, 49]. It will be the main focus of our study.

The intuitive understanding is that the system (model, field theory) is integrable if all physical observables for this system can be, in principle, computed exactly, without making any approximations. In practice the answer for those observables is rarely obtained in closed analytical form. So typically we call the model integrable when we know the trick that, without any approximation or guesswork, reduces the dynamical problem to much simpler static one which can be solved numerically with arbitrary precision in reasonable amount of time. The distinctive feature of integrable quantum field theories is the existence of infinite number of independent commuting integrals of motion.

In the original paper [48] the Bukhvostov-Lipatov model was shown in the original paper to be integrable by coordinate Bethe ansatz. This technique allows one to construct the eigenstates of the Hamiltonian as a combination of elementary excitations, with momenta determined by a coupled set of algebraic equations. In this description the vacuum energy is just a sum of energies of quasiparticles filling the Dirac sea. However, since the energy of one excitation is negative, the ground state is made of infinite number of such quasiparticles, and Bethe ansatz equations do not make sense without some regularization. Moreover, even regularized versions are notoriously difficult to treat without a prior knowledge of some approximation to the solution.

Some light can be cast by noting that Bethe ansatz equations can be represented as a finite set of functional relations combined with some assumptions about analytical properties of involved functions. For several integrable models same relations were implemented as relations between monodromy data of some family of ODEs, with Bethe roots corresponding to zeroes of connection coefficients as functions of spectral parameter. That approach, known as the ODE/IM correspondence [51, 52] (the abbreviation stands for "*Ordinary Differential Equation/ Integrals of Motion*"), gives several insights into problem. Firstly, asymptotical behaviour of connection coefficients can be investigated using WKB approach - an expansion similar to quasiclassical expansion in quantum mechanics. Secondly, it is easier to seek approximate positions of individual roots by solving ODE numerically, because unlike Bethe ansatz equations the ODE allows one to search for individual zeroes separately instead of trying to solve for all infinite set of roots at once. In subsequent work [61] the correspondence was upgraded to ODE/IQFT: expectation values of integrals of motion of the quantum field theory were related to the conserved quantities of an integrable PDE. This observation allowed to apply powerful technique known as the classical inverse scattering method [53] to the problem of computation of the quantum observables.

The ODE/IM correspondence was studied on several examples [54–64], but its origin remained a mystery. In this work I prove the ODE/IM correspondence for Bukhvostov-Lipatov model and apply it to compute the vacuum energy of the model. Moreover, it will be demonstrated that not just some observables, but the whole quantum state of Bukhvostov-Lipatov model is in some sense encoded into the solution of an integrable PDE, and thus two systems are related by *Quantum/Classical* *duality*. Note that the PDE in question, so-called modified sinh-Gordon equation, is not a classical limit of Bukhvostov-Lipatov model, but represents an entirely different classical field theory.

This thesis gives complete and systematical presentation of ideas and results of our works [65, 66] where all principal results of this thesis were reported. In this thesis we provide extensive derivations of all results.

The structure of this work is as follows. In chapter 2 Bukhvostov-Lipatov model is introduced and a brief review of results by Bukhvostov and Lipatov is given. In chapters 3 and 4 I derive two limits for scaling function of the model using perturbation theory. Chapter 5 deals with derivation and regularization of Bethe ansatz equations for Bukhvostov-Lipatov model. Chapter 6 gives review of ODE/IM correspondence and provides the proof of ODE/IM correspondence for Bukhvostov-Lipatov model, and in chapter 7 we derive non-linear integral equations for vacuum state and an exact formula for vacuum energy.

Chapter 2

Instanton counting in $O(3)$ non-linear sigma model

2.1 $O(3)$ NLSM. An overview

A sigma model is a field theory which describes embedding of one manifold (called "worldsheet", or "coordinate space") into another ("target space"), where coordinates on target space are treated as bosonic fields. For $O(3)$ non-linear sigma model the worldsheet is 2-dimensional plane with Euclidean (or Minkowski) metric, and the target space is $\mathbb{S}^2$. Its Lagrangian can be written down as

$$\mathcal{L} = \frac{1}{2g}(\partial_\mu n_i)^2 \qquad \sum_i n_i^2 = 1 \tag{2.1}$$

The constraint on bosonic fields n_i, which are dimensionless components of a 3-vector, introduces non-linear interaction. At the classical level this model is conformally invariant. This is not the case in quantum version of the model.

Even though naive perturbative analysis leads to the conclusion that $O(N)$ sigma models should have $N-1$ massless excitations, non-perturbative arguments [43–46, 67] imply that such models have mass gap. This phenomenon was dubbed "dimensional transmutation", as dimensionful parameter was introduced into the system by quantum corrections. It is well known that bosonic part of QCD, Yang-Mills theory with gauge group $SU(3)$, behaves the same way.

2.2 Instantons in QM and QFT

This section provides a short introduction into instantons on example of quantum mechanics in one dimension. It loosely follows discussion in [68] and is presented here for reader's convenience.

The principal approach to problems involving nontrivial potentials in quantum mechanics is perturbation theory. If the problem can be represented as a slight perturbation of another one we know how to solve, we can take solution to the latter as a zero-order approximation and compute corrections to it as a series in small deformation parameter. There are few potentials in one-dimensional quantum mechanics which can be solved explicitly and the most common of them is the potential of harmonic oscillator. Typical application of perturbation theory looks like this: we find a local minimum to our potential and approximate it with a quadratic potential with the same minimum. Then wavefunctions would be close to eigenfunctions of harmonic oscillator as long as approximation is valid. However, this approach leads us astray when the minimum is not the only global one. Consider double well potential

$$V(x) = \lambda\left(x^2 - \eta^2\right)^2 \tag{2.2}$$

This potential near minima can be approximated by harmonic oscillator potential

$$V(x) \approx \tfrac{1}{2}\omega^2(x \pm \eta)^2\,, \qquad \omega^2 = 8\lambda\eta^2 \tag{2.3}$$

Then we would expect that solution is centered in one of the wells

$$\langle x\rangle = \pm\eta + \text{small perturbative corrections} \tag{2.4}$$

and that the ground state is degenerate due to the symmetry.

However, exact symmetry considerations imply that true ground state is described by symmetric wavefunction and is not degenerate. Indeed, more accurate analysis shows that

$$E_0 = \frac{\omega}{2}\left(1 - \sqrt{\frac{2\omega^3}{\pi\lambda}}\mathsf{e}^{-\frac{\omega^3}{12\lambda}}\right) \tag{2.5}$$

$$E_1 = \frac{\omega}{2}\left(1 + \sqrt{\frac{2\omega^3}{\pi\lambda}}\mathsf{e}^{-\frac{\omega^3}{12\lambda}}\right) \tag{2.6}$$

provided $\omega^3 \gg \lambda$ This simple exercise demonstrates that at certain conditions naive perturbative considerations lead to qualitatively wrong answers. Moreover we can notice that corrections to the ground state energy that lift the degeneracy are non-perturbative, i.e. they equal to zero in all orders in λ.

From physical point of view the solution localised in one well can not be a stable one because quantum particles are capable of quantum tunneling - a transition through classically forbidden zone. To describe such processes it is useful to perform so-called Wick rotation, an analytical continuation of time axis in imaginary direction. Then

$$t = \mathrm{i}t_E\,, \qquad t_E \in \mathbb{R} \tag{2.7}$$

and we can introduce Euclidean action given by

$$S_E = \int \mathcal{L}_E(t_E)\mathsf{d}t_E = -\int \left(\tfrac{1}{2}m\dot{x}^2 + V(x)\right)\mathsf{d}t_E \tag{2.8}$$

where the dot stands for derivative w.r.t. Euclidean time t_E. It can be interpreted as an analytical continuation of usual action S. But we can also interpret this functional as normal action for a system with potential

$$U(x) = -V(x) \tag{2.9}$$

This new system admits classical solutions minimizing S_E which are solutions to Euclidean equations of motion.

$$m\ddot{x} = +\frac{\mathsf{d}V}{\mathsf{d}x} \tag{2.10}$$

If we impose boundary conditions

$$x(-\infty) = -\eta\,, \qquad x(+\infty) = \eta \tag{2.11}$$

a family of solutions labeled by an arbitrary real parameter t_c can be found explicitly:

$$x(t_E) = \eta \tanh\frac{\omega(t_E - t_c)}{2} \tag{2.12}$$

Solution (2.12) is the simplest example of *instanton* - a solution to Euclidean equation of motion interpolating between minima of potential. Such solutions appear in systems where classical minimum of potential is degenerate, in particular in some quantum field theories. The presence of instantons affects low energy behaviour of such systems, in particular the properties of ground state. In context of field theories we shall refer to minima of potentials as to *classical vacua*, in contrast with *quantum vacuum* (exact ground state of the system). If we use path integral formalism, then vacuum expectation values of observables can be computed as sums over trajectories

$$\langle\mathcal{O}\rangle = Z^{-1}\int \mathcal{D}[x(t)]\mathcal{O}\mathsf{e}^{-S_E[x(t)]} \tag{2.13}$$

Then it makes sense to speak about contributions of a trajectory to the expectation values of observables. Solutions that minimize Euclidean action obviously give the largest contributions. Even though such solutions typically constitute a set of measure zero, small oscillations near those stationary points of Euclidean action give comparable contributions and must be taken into account. Perturbative series are generated by just one stationary point. In the double well example above (2.2) trajectories contributing to perturbative series are small oscillations around

$$x(t_E) \equiv -\eta \tag{2.14}$$

(naive symmetrization adds trajectories in vicinity of $x(t_E) \equiv +\eta$ as well). Trajectories close to the instanton do not appear in perturbative series, but can give significant contributions to the expectation values of observables. For example, the leading correction to the ground state energy in double well potential (2.2) is defined by Euclidean action of instanton (2.12)

$$\Delta E = \frac{\omega}{2}\sqrt{\frac{2\omega^3}{\pi\lambda}}\mathsf{e}^{-\frac{\omega^3}{12\lambda}} \sim \mathsf{e}^{-S_{inst}}\,, \qquad S_{inst} = \frac{\omega^3}{12\lambda} \tag{2.15}$$

2.3 Instantons in $O(3)$ NLSM

In this section I review the derivation of partition function of instantons in "instanton gas" approximation. The discussion loosely follows [47] and [48].

2.3.1 The instanton solution

The sigma model has infinite number of classical vacua: arbitrary constant value of field minimises the energy. So the moduli space of vacua is a two-dimensional sphere. Every pair of vacua is connected by instanton solutions [69]. Since spacetime can be compactified to a sphere, and its fundamental group is $\pi_2(\mathbb{S}^2) = \mathbb{Z}$, any configuration can be assigned to a topological class labeled by an integer number which tells how many times the end of unit vector representing field at some arbitrary point of space passes around n_3 axis while time changes from $-\infty$ to ∞. It is useful to introduce complex coordinates both on worldsheet and in target space via stereographic projection:

$$z = x + \mathrm{i}t, \qquad v = \frac{n_1 + \mathrm{i}n_2}{1 - n_3} \tag{2.16}$$

Then action assumes the following form:

$$S = \frac{4}{g}\int \frac{\mathsf{d}^2 z}{(1+|v|^2)^2}\left(\frac{\partial v}{\partial z}\frac{\partial \bar{v}}{\partial \bar{z}} + \frac{\partial v}{\partial \bar{z}}\frac{\partial \bar{v}}{\partial z}\right)\,. \tag{2.17}$$

Therefore the Euclidean equation of motion reads

$$\partial_z \partial_{\bar{z}} v(z, \bar{z}) = 0 \tag{2.18}$$

and solutions are purely holomorphic or antiholomorphic. All configurations are maps between two-dimensional spheres, and thus can be split into discrete classes labeled by its index, or winding number. We shall also use the term "topological charge" sometimes. The winding number has an integral representation which reads

$$k = \frac{1}{\pi}\int \frac{\mathsf{d}^2 z}{(1+|v|^2)^2}\left(\frac{\partial v}{\partial z}\frac{\partial \bar{v}}{\partial \bar{z}} - \frac{\partial v}{\partial \bar{z}}\frac{\partial \bar{v}}{\partial z}\right) \tag{2.19}$$

In terms of the original fields n_i it can be expressed as

$$k = \frac{1}{\pi}\int \frac{\mathsf{d}\,x\,\mathsf{d}\,t}{1-n_3}\begin{vmatrix} n_1 & n_2 & 1-n_3 \\ \partial_x n_1 & \partial_x n_2 & -\partial_x n_3 \\ \partial_t n_1 & \partial_t n_2 & -\partial_t n_3 \end{vmatrix} \tag{2.20}$$

The majority of solutions (i.e. all solutions except a subset of measure zero) can be represented in one of the two forms listed below depending on the sign of winding number.

$$v_+(z) = h\prod_{i=1}^{k}\frac{z-a_i}{z-b_i} \quad k \geq 0 \tag{2.21}$$

$$v_-(z) = h\prod_{i=1}^{-k}\frac{\bar{z}-a_i}{\bar{z}-b_i} \quad k \leq 0 \tag{2.22}$$

We shall call solutions with positive winding number "instantons", and those with negative winding number "anti-instantons"

2.3.2 Partition function of instantons

In the language of path integral, perturbation theory allows to compute contribution of closed trajectories oscillating in vicinity of only one preselected classical vacuum, while instantons connect different vacua. There are also trajectories that oscillate around instantons, which can be thought of as perturbative expansion around instantons. Thus in order to compute contribution of instantons to partition function one needs to take these trajectories into account. Action can be explicitly split into topological charge and non-negative part:

$$S = \frac{4\pi k}{g} + \frac{8}{g}\int \mathsf{d}^2 z(1+|v|^2)^{-2}\left|\frac{\partial v}{\partial \bar{z}}\right|^2 \tag{2.23}$$

Let us consider configurations that only slightly differ from some instanton solution v_k:

$$v(z,\bar{z}) = v_k(z) + \nu(z,\bar{z}) \tag{2.24}$$

Then action of such configuration can be approximately represented as

$$S = \frac{4\pi k}{g} + \frac{8}{g}\int \mathsf{d}^2 z (1+|v|^2)^{-2} |\partial_{\bar{z}}\nu|^2 \tag{2.25}$$

Here we have neglected corrections that are at least cubic in ν. In this approximation the problem reduces to computation of determinant of certain Laplacian-like differential operator. These determinants are UV-divergent and therefore some regularization is required. The divergence in question can be interpreted as a contribution of small-size instantons. The computation of determinants was first performed by Fateev, Frolov and Schwartz [47]. They have shown that an arbitrary correlation function of $O(3)$ NLSM in considered approximation

$$I(\phi) = \int \mathcal{D}v\, \phi[v]\, \mathsf{e}^{-S[v]} \tag{2.26}$$

can be represented as a correlation function in classical Coulomb system of temperature $T = 1$. In particular, the partition function

$$Z_{inst} = \Xi = \sum_q \frac{K^q}{q!^2} \int \exp\left(-\frac{1}{T}\epsilon_q(a,b)\right) \prod_j \mathsf{d}^2 a_j d^2 b_j \tag{2.27}$$

with

$$\epsilon_q(a,b) = -\sum_{i<j}^{q} \log|a_i - a_j|^2 - \sum_{i<j}^{q} \log|b_i - b_j|^2 + \sum_{i,j} \log|a_i - b_j|^2 \tag{2.28}$$

and K being renormalization-dependent constant. Furthermore, since this temperature is above critical temperature, the Coulomb gaz is in deconfinement phase, i.e. distance between two particles of opposite charge is not significantly smaller than distance between particles of same charge. It means that there is a Debyi screening, i.e. a finite correlation length is introduced into the system. Inverse correlation length depends on the UV cutoff and physical coupling:

$$m = C\frac{\Lambda}{g_{phys}} \mathsf{e}^{-\frac{2\pi}{g_{phys}}} \tag{2.29}$$

2.3.3 Instanton–anti-instanton interaction

There are two reasons [48] to consider instanton–anti-instanton (i-a) interaction. Firstly, expansion near instantons is semiclassical one, and by integrating quadratic fluctuations near instantons we compute first non-trivial order of this expansion. However, fluctuations near trajectories that go back and forth between several vacua should have the same order, even though such trajectories are not solutions to classical equations of motion. Secondly, by adding partition functions of instantons and anti-instantons we take into account sector $k = 0$ twice. Subtracting it is clearly not the whole answer as contribution from this sector is not extensive, while both instanton partition function and the whole answer must be extensive.

The energy of i-a interaction is defined as difference

$$S_{int} = S - S_i - S_a \tag{2.30}$$

To compute it we use trial function

$$w = h \prod_{i,j} \frac{z - a_i}{z - b_i} \frac{\bar{z} - \bar{c}_j}{\bar{z} - \bar{d}_j} \tag{2.31}$$

which is close enough to a solution provided

$$|a_i - b_i| << |a_i - c_j| \tag{2.32}$$

Indeed, when we are far from poles of anti-instanton part of trial function, it is almost a constant, and w is roughly an istanton. The condition (2.32) is satisfied if instantons and anti-instantons are localised in clusters such that their size is significantly less than the separation between clusters.

It is convenient to perform a change of variables $w = e^u$ so that action takes form

$$S = \int \mathrm{d}^2 x \, \mathcal{L} \qquad \mathcal{L} = \frac{1}{g} \frac{1}{\cosh^2 \frac{u+\bar{u}}{2}} \left(\frac{\partial u}{\partial z} \frac{\partial \bar{u}}{\partial \bar{z}} - \frac{\partial u}{\partial \bar{z}} \frac{\partial \bar{u}}{\partial z} \right) \tag{2.33}$$

We introduce notations u_1, u_2, A for approximate solutions in the cluster of instantons, cluster of anti-instantons and the rest of the space respectively:

$$u = \log w = u_1 + u_2 + A \tag{2.34}$$

$$u_1 = \log \prod_{i=1}^{n_1} \frac{z - a_i}{z - b_i} \frac{D - b_i}{D - a_i} \qquad u_2 = \log \prod_{i=1}^{n_2} \frac{\bar{z} - \bar{c}_i}{\bar{z} - \bar{d}_i} \frac{\bar{B} - \bar{c}_i}{\bar{B} - \bar{d}_i} \tag{2.35}$$

$$A = \log \prod_{i=1}^{n_1} \frac{D - a_i}{D - b_i} \prod_{j=1}^{n_2} \frac{\bar{B} - \bar{c}_j}{\bar{B} - \bar{d}_j} \tag{2.36}$$

The energy of interaction then reads

$$S_{int} = \frac{16\pi h^2}{g(1+h^2)^2} \sum_{i,j} \log \left(\frac{|a_i - c_j||b_i - d_j|}{|a_i - d_j||b_i - c_j|} \right) \tag{2.37}$$

This action is still quite complicated, but it can be further simplified by introducing phenomenological constant f_1 which represents the result of averaging over h:

$$\frac{16\pi h^2}{g(1+h^2)^2} = 2f_1 \tag{2.38}$$

This way the partition function of interacting instantons and anti-instantons can be represented as a partition function of Coulomb gas with two kinds of particles and two constants of interaction:

$$Z = \sum_{n_1,n_2} (n_1!)^{-2}(n_2!)^{-2}\left(\frac{m}{2\pi}\right)^{2(n_1+n_2)} \int \prod_{i,j} \mathsf{d}^2 a_i \mathsf{d}^2 b_i \mathsf{d}^2 c_j \mathsf{d}^2 d_j \mathsf{e}^{-U} \tag{2.39}$$

$$\begin{aligned} U = -f_0 \Bigg[&\sum_{i<i'} \left(\log|a_i - a_{i'}|^2 + \log|b_i - b_{i'}|^2\right) + \sum_{j<j'} \left(\log|c_j - c_{j'}|^2 + \log|d_j - d_{j'}|^2\right) \\ &- \sum_{i,i'} \log|a_i - b_{i'}|^2 - \sum_{j,j'} \log|c_j - d_{j'}|^2 \Bigg] \\ &- f_1 \sum_{i<j} \left(\log|a_i - c_j|^2 + \log|b_i - d_j|^2 - \log|a_i - d_j|^2 - \log|b_i - c_j|^2\right) \end{aligned} \tag{2.40}$$

Constant f_0 in last expression appears as an attempt to take into account quantum corrections to temperature appearing due to the new interaction. Bukhvostov and Lipatov have demonstrated that this partition function is equivalent to the partition function of a QFT with two massive scalar bosonic fields:

$$Z = \int \mathcal{D}\phi \exp\left(-\int \left[\tfrac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \tfrac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 + \frac{\mu}{2\pi}\cos(\lambda_1\phi_1)\cos(\lambda_2\phi_2)\right]\right) \tag{2.41}$$

Constants λ_1,λ_2 and μ all depend on cutoff. Note that, despite classical intuition, dimension of μ in quantum field theory depends on the couplings λ_1, λ_2. In particular if $\lambda_1^2 + \lambda_2^2 = 4\pi$ parameter μ has the dimension of mass (see discussion in chapter 4). The theory governed by Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \tfrac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 + \frac{\mu}{\pi}\cos(\lambda_1\phi_1)\cos(\lambda_2\phi_2) \tag{2.42}$$

is integrable provided

$$\lambda_1^2 + \lambda_2^2 = 4\pi\,. \tag{2.43}$$

In this case it is convenient to parametrize couplings as

$$\lambda_i = \sqrt{2\pi a_i}\,, \qquad a_1 = 1-\delta, a_2 = 1+\delta\,, 0 \le \delta \le 1 \tag{2.44}$$

Note that in literature, including the original work [48], one can find different prescription:

$$\frac{\pi}{\lambda_1^2} + \frac{\pi}{\lambda_2^2} = 1 \tag{2.45}$$

This contradiction is resolved if one notes that dual fermionic Lagrangian obtained in case (2.45) via boson-fermion duality would describe a theory integrable by coordinate Bethe ansatz if interpreted as a Lagrangian of *bare* theory. However, the boson-fermion duality actually relates *renormalized* theories. It is possible to choose regularization scheme such that fermionic Lagrangian obtained from (2.43) is a renormalized Lagrangian (including all necessary counterterms) obtained from an integrable bare theory.

2.3.4 Bukhvostov-Lipatov model

The bosonic QFT (2.42) can be rewritten as an equivalent fermionic model. Firstly consider change of variables

$$\chi_\pm(x) = \frac{1}{\sqrt{\pi}} \left(\lambda_1 \phi_1(x) \pm \lambda_2 \phi_2(x)\right) \tag{2.46}$$

In terms of new bosonic fields $\chi_\pm$ Lagrangian reads

$$\begin{aligned} \mathcal{L} = \left(\frac{\pi}{\lambda_1^2} + \frac{\pi}{\lambda_2^2}\right) \left(\partial_\mu \chi_+)^2 + (\partial_\mu \chi_-)^2\right) + \pi \left(\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2}\right) \partial_\mu \chi_+ \partial^\mu \chi_- \\ + \frac{\mu}{4\pi} \cos(\sqrt{4\pi}\chi_+) + \frac{\mu}{4\pi} \cos(\sqrt{4\pi}\chi_-) \end{aligned} \tag{2.47}$$

Resulting Lagrangian is nothing but two copies of Lagrangian of sine-Gordon theory plus mixed kinetic term. It is possible to obtain equivalent fermionic Lagrangian for this theory similarly to Coleman-Fröhlich relations for sine-Gordon model [70–72]. Consider soliton creation operators

$$\psi_{\sigma,s}(x) = C : \exp\left(-\frac{\mathrm{i}}{\sqrt{\pi}} \int\limits_0^x \dot{\chi}_\sigma(y) dy + 2s\mathrm{i}\sqrt{\pi}\chi_\sigma\right) : \qquad s = \pm 1\,, \qquad \sigma = \pm \tag{2.48}$$

with some normalization factor C. It is easy to check that they can be normalized to satisfy standard anticommutation relations

$$\{\psi_{\sigma,s}(x), \psi_{\sigma',s'}(y)\} = 0\,, \qquad \{\psi^\dagger_{\sigma,s}(x), \psi_{\sigma',s'}(y)\} = \delta_{ss'}\delta_{\sigma\sigma'}\delta(x-y) \tag{2.49}$$

and thus represent local fermionic fields. In order to construct Lagrangian the following identification (consistent with (2.48)) is needed:

$$\partial_\mu \chi_\sigma = \bar{\Psi}_\sigma \gamma^\mu \Psi_\sigma\,, \qquad \frac{\mu}{4\pi} \cos(\sqrt{4\pi}\chi_\sigma) = M\bar{\Psi}\Psi \tag{2.50}$$

Field Ψ is a two-component spinor

$$\Psi_\sigma = \begin{pmatrix} \psi_{\sigma,+} \\ \psi_{\sigma,-} \end{pmatrix} \tag{2.51}$$

As a result we obtain the following Lagrangian:

$$\mathcal{L} = \sum_{a=\pm} \bar{\Psi}_a (\mathrm{i}\gamma^\mu \partial_\mu - M)\Psi_a - g(\bar{\Psi}_+\gamma_\mu\Psi_+)(\bar{\Psi}_-\gamma^\mu\Psi_-) - \sum_a \frac{g_1}{2}(\bar{\Psi}_a\gamma_\mu\Psi_a)^2 \tag{2.52}$$

In integrable case (2.43) coupling constant g of the dual model reads

$$g = \frac{\pi\delta}{1-\delta^2} \tag{2.53}$$

2.4 Fateev Model. Exact instanton counting

The Bukhvostov-Lipatov model is a special case of Fateev model [73] described by the following Lagrangian:

$$\mathcal{L}_F = \frac{1}{16\pi}\sum_{i=1}^{3}(\partial_\mu\phi_i)^2 + 2\mu \mathrm{e}^{\mathrm{i}\alpha_3\phi_3}\cos(\alpha_1\phi_1+\alpha_2\phi_2) + 2\mu\mathrm{e}^{-\mathrm{i}\alpha_3\phi_3}\cos(\alpha_1\phi_1-\alpha_2\phi_2) \tag{2.54}$$

It is also integrable provided

$$a_1 + a_2 + a_3 = 2\,, \qquad a_i = 4\alpha_i^2 \tag{2.55}$$

The model (2.54) also admits a dual description as a sigma model which can be considered as an integrable deformation of the $O(4)$ NLSM.

The Fateev model was studied extensively in the literature. In particular, Fateev himself has found the factorized S-matrix and computed large-R asymptotics of the ground state energy using thermodynamic Bethe ansatz approach. He has also shown that the model is UV-finite. The ODE/IM (ODE/IQFT) construction, which in principle allows exact computation of integrals of motion beyond weak coupling limit, was considered for various regions of parameter space of the model by Lukyanov, Bazhanov and Kotousov in a series of works [74–79].

It was thoroughly studied for the case $a_1 > 0, a_2 > 0, a_3 > 0$ in [77], in which the finite system of nonlinear integral equations and integral representations for the vacuum expectation values of quantum integrals of motion were derived. The answers of [77] were checked against conformal limit, but an independent proof for arbitrary R, M, δ was not found. Moreover, there was no formal justification for the use of the ODE/IM construction in this case. Strictly speaking, this result at the time was a conjecture. The Bukhvostov-Lipatov case is special because it can be solved exactly using coordinate Bethe ansatz, which allows to prove the ODE/IQFT correspondence explicitly. This proof was obtained in works [65, 66] by Bazhanov, Lukyanov and the author. It is also the primary focus of the present dissertation.

The case $a_1 < 0$, $a_2 > 0$, $a_3 = 0$ was considered by Bazhanov, Lukyanov and Kotousov [78,79]. The latter is interesting as it represents so-called "sausage" sigma model [80], which is an integrable deformation of the $O(3)$ NLSM. It does not have fermionic form, but it can be described by an analytical continuation of the bosonic Lagrangian (2.42) into the region $\delta > 1$. In the limit $\delta \to \infty$ the full original $O(3)$ NLSM is restored. This observation vindicates the approximations made during the derivation of Bukhvostov-Lipatov model. Moreover, it introduces the idea of exact instanton counting: summing contributions of instantons and instanton–anti-instanton configurations gives us the exact answer for full partition function including the perturbative part.

Weak coupling expansion. Renormalised Perturbation Theory.

The most straightforward way to compute vacuum energy of Bukhvostov-Lipatov model is the perturbation theory, i.e. expansion in powers of small coupling constant g. Most general Lagrangian consistent with $U(1) \times U(1)$ symmetry is

$$\mathcal{L} = \sum_{a=\pm} \bar{\Psi}_a(\mathrm{i}\gamma^\mu \partial_\mu - M)\Psi_a - g(\bar{\Psi}_+\gamma_\mu\Psi_+)(\bar{\Psi}_-\gamma^\mu\Psi_-) - \sum_a \frac{g_1}{2}(\bar{\Psi}_a\gamma_\mu\Psi_a)^2 \tag{3.1}$$

Integrable case corresponds to vanishing physical coupling g_1. This theory contains logarithmic UV divergences. It can be regularized by adding counterterms

$$\delta\mathcal{L} = -\sum_{a=\pm}\left(\delta M \bar{\Psi}_a\Psi_a + \frac{g_1^{(c)}}{2}(\bar{\Psi}_a\gamma_\mu\Psi_a)^2\right) \tag{3.2}$$

Parameters $g_1^{(c)}$ and δM are not independent. In fact, it is possible to choose renormalization scheme such that

$$\delta M = 0\,, \qquad g_1^{(c)} = \frac{g^2}{2\pi} \tag{3.3}$$

Throughout this chapter we will use the notation g_1 for counterterm coupling, as physical coupling is zero. To sum up, renormalized Lagrangian reads

$$\mathcal{L} = \sum_{a=\pm} \bar{\Psi}_a(\mathrm{i}\gamma^\mu \partial_\mu - M)\Psi_a - g(\bar{\Psi}_+\gamma_\mu\Psi_+)(\bar{\Psi}_-\gamma^\mu\Psi_-) - \sum_a \frac{g^2}{4\pi}(\bar{\Psi}_a\gamma_\mu\Psi_a)^2 \tag{3.4}$$

The main goal of this chapter is to compute the scaling function

$$\mathfrak{F} = \frac{R}{\pi}\left(E - \mathcal{E}R\right) \tag{3.5}$$

in two-loop approximation:

$$\mathfrak{F} = \mathfrak{F}_0 + \frac{g}{\pi}\mathfrak{F}_1 + \frac{g^2}{\pi^2}\mathfrak{F}_2 + o(g^2) \tag{3.6}$$

This computation allows to establish asymptotics of vacuum energy for large values of $r = MR$ against which we shall compare our exact results later.

3.1 Specific bulk energy from Fateev model

The expression (3.5) for scaling function involves the specific bulk energy $\mathcal{E}$. For Bukhvostov-Lipatov model it is divergent. However, to simplify the computation of scaling function it is useful to find the structure of divergences which should match divergences of perturbative part. Fateev model (2.54) can be considered as an integrable generalization of Bukhvostov-Lipatov model. In the limit

$$\nu = \tfrac{1}{2}a_3 = 1 - \tfrac{1}{2}a_1 - \tfrac{1}{2}a_2 \to 0 \tag{3.7}$$

third bosonic field decouples and becomes free. In his work Fateev has computed specific bulk energy of the Fateev model explicitely:

$$\mathcal{E}_F = -\pi\mu^2 \prod_{i=1}^{3} \frac{\Gamma\left(\frac{a_i}{2}\right)}{\Gamma\left(1-\frac{a_i}{2}\right)} \tag{3.8}$$

This expression becomes divergent at $\nu = 0$:

$$\mathcal{E}_F = \pi\mu^2\left(-\frac{2}{a_3} - 4\log 2 + \psi\left(\frac{1-\delta}{2}\right) + \psi\left(\frac{1+\delta}{2}\right) - 2\psi(\tfrac{1}{2}) + o(1)\right) \tag{3.9}$$

Obviously we expect that

$$\mathcal{E}_F \to \mathcal{E} + const\ m^2 \tag{3.10}$$

where $\mathcal{E}$ stands for the specific bulk energy of the Bukhvostov-Lipatov model, and term proportional to m^2 represents the contribution of the free boson. Considering Fateev model as an analytical regularization of Bukhvostov-Lipatov model, and taking into account the fact that a_3^{-1} corresponds to $-\log(\mu\epsilon)$ in short distance regularization we can write specific bulk energy of Bukhvostov-Lipatov model as

$$\mathcal{E} = w(g)\,\epsilon^{-2} + \frac{M^2}{\pi}\cos^2\left(\tfrac{\pi\delta}{2}\right)\ \log\big(M\epsilon\big) + o(1)\ , \tag{3.11}$$

where $w(g)$ is some nonuniversal function of coupling. Note that the coefficient in front of the logarithm is universal.

3.2 Matsubara propagator

Since our goal is to compute vacuum energy in finite volume (i.e. with compactified spatial dimension), we need to modify our diagram technique accordingly. We shall use Matsubara technique with "temperature" R^{-1} and "chemical potential" $\mathsf{e}^{2\pi \mathrm{i} k_\pm}$. The propagator

$$\mathbf{S}_\sigma(\mathbf{x}) = \langle \Psi_\sigma(\mathbf{x}) \otimes \bar{\Psi}_\sigma(0) \rangle \tag{3.12}$$

can be represented as

$$\mathbf{S}_\sigma(\mathbf{x}) = (M - \gamma^\mu \partial_\mu)\, G_\sigma(\mathbf{x}) \tag{3.13}$$

Note that as spatial dimension is compactified, spatial part of momentum is quantised:

$$G_\sigma(x) = \frac{1}{R} \sum_{n=-\infty}^{\infty} \mathsf{e}^{\mathrm{i}\pi(2n+1+2k_\sigma)x/R} \int_{-\infty}^{\infty} \frac{\mathsf{d}\omega}{2\pi} \frac{\mathsf{e}^{\mathrm{i}\omega t}}{\omega^2 + \frac{\pi^2}{R^2}(2n+1+2k_\sigma)^2 + M^2} \tag{3.14}$$

In this section we shall use Euclidean metric and Euclidean gamma matrices unless otherwise specified. To compute the sum we use Poisson summation formula

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \hat{f}(2\pi k) \tag{3.15}$$

where $\hat{f}$ stands for Fourier transform of f. We choose function f_σ as Fourier transform of function $G_\sigma(\mathbf{x})$ w.r.t. temporal coordinate t

$$f_\sigma(y) = \frac{\mathsf{e}^{\mathrm{i}\pi(2y+1+2k_\sigma)Mx/r}}{\frac{\omega^2}{M^2} + \frac{\pi^2}{r^2}(2y+1+2k_\sigma)^2 + 1} \tag{3.16}$$

so that $G(x)$ can be represented as

$$G_\sigma(\mathbf{x}) = \frac{1}{2\pi M^2 R} \int_{-\infty}^{\infty} \mathsf{d}\omega\, \mathsf{e}^{-\mathrm{i}\omega t} \sum_{n=-\infty}^{\infty} f_\sigma(n) \tag{3.17}$$

Introducing dimensionless coordinates w,ϕ such that

$$w = M|t + \mathrm{i}x| \qquad Mt = w\cos\phi \qquad Mx = w\sin\phi \tag{3.18}$$

we express $\hat{f}$ as

$$\hat{f}_\sigma(q) = \int \mathsf{d}y \frac{\mathsf{e}^{\mathrm{i}qy + \mathrm{i}\sin\phi\,\pi(2y+1+2k_\sigma)w/r}}{\frac{\omega^2}{M^2} + \frac{\pi^2}{r^2}(2y+1+2k_\sigma)^2 + 1} \tag{3.19}$$

This integral is convergent and can be computed by residues: for $q > -\frac{2\pi w \sin\phi}{r}$ integral evaluates to

$$\hat{f}_\sigma(q) = r\mathsf{e}^{-M^{-1}\sqrt{\omega^2+M^2}(\frac{qr}{2\pi}+w\sin\phi)}\frac{M\mathsf{e}^{-\mathrm{i}q(k_\sigma+\frac{1}{2})}}{2\sqrt{\omega^2+M^2}} \tag{3.20}$$

Otherwise result reads

$$\hat{f}_\sigma(q) = r\mathsf{e}^{M^{-1}\sqrt{\omega^2+M^2}(\frac{qr}{2\pi}+w\sin\phi)}\frac{M\mathsf{e}^{-\mathrm{i}q(k_\sigma+\frac{1}{2})}}{2\sqrt{\omega^2+M^2}} \tag{3.21}$$

Now sum over values of q that are integer multiple of 2π can be easily computed as a sum of geometric progression. Let $n_0 = 0$ be lowest value of q for which contour is closed in the upper half plane. Depending on spatial coordinate, $-1 \le n_0 \le 0$. Then

$$\sum_{n=0}^{\infty}\hat{f}_\sigma(2\pi n) = \frac{rM\mathsf{e}^{-wM^{-1}\sin\phi\sqrt{\omega^2+M^2}}}{2\sqrt{\omega^2+M^2}}\frac{1}{1+\mathsf{e}^{-rM^{-1}\sqrt{\omega^2+M^2}-2\pi\mathrm{i}k_\sigma}} \tag{3.22}$$

$$\sum_{n=-\infty}^{-1}\hat{f}_\sigma(2\pi n) = -\frac{rM\mathsf{e}^{wM^{-1}\sin\phi\sqrt{\omega^2+M^2}}}{2\sqrt{\omega^2+M^2}}\frac{\mathsf{e}^{-rM^{-1}\sqrt{\omega^2+M^2}+2\pi\mathrm{i}k_\sigma}}{1+\mathsf{e}^{-rM^{-1}\sqrt{\omega^2+M^2}+2\pi\mathrm{i}k_\sigma}} \tag{3.23}$$

Now let us perform integration over ω. Introducing variable θ related to ω by

$$\frac{\omega}{M} = \sinh\theta \quad \sqrt{\frac{\omega^2}{M^2}+1} = \cosh\theta \tag{3.24}$$

we get

$$\frac{1}{2\pi R}\int_{-\infty}^{\infty}\mathsf{d}\omega\sum_{n=0}^{\infty}\hat{f}_\sigma(2\pi n) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\mathsf{d}\theta\frac{\mathsf{e}^{\mathrm{i}w\sinh(\theta+\mathrm{i}\phi)}}{1+\mathsf{e}^{-r\cosh\theta-2\pi\mathrm{i}k_\sigma}} \tag{3.25}$$

Now we expand denominator in Taylor series and make use of integral representation of Macdonald function K_0. Introducing notation ϕ' by $x+nr = |t+\mathrm{i}(x+nr)|\sin\phi'$ we get

$$\begin{aligned}\frac{1}{2\pi R}\int_{-\infty}^{\infty}\mathsf{d}\omega\sum_{n=0}^{\infty}\hat{f}_\sigma(2\pi n) =& \frac{1}{4\pi}\sum_{n=0}^{\infty}(-1)^n\mathsf{e}^{-2\pi\mathrm{i}k_\sigma n}\int_{-\infty}^{\infty}\mathsf{e}^{-M|t+\mathrm{i}x+\mathrm{i}nr|\cosh(\theta+\mathrm{i}\phi')}\mathsf{d}\theta\\ =&\frac{1}{2\pi}\sum_{n=0}^{\infty}(-1)^n\mathsf{e}^{-2\pi\mathrm{i}k_\sigma n}K_0(M|t+\mathrm{i}x+\mathrm{i}nr|)\end{aligned} \tag{3.26}$$

Similarly, computing Fourier transform of (3.23) we obtain

$$\frac{1}{2\pi R}\int\limits_{-\infty}^{\infty}\mathsf{d}\omega\sum_{n=-\infty}^{-1}\hat{f}_\sigma(2\pi n)=\frac{1}{2\pi}\sum_{n=1}^{\infty}(-1)^n\mathsf{e}^{2\pi\mathrm{i}k_\sigma n}K_0(M|t-ix+inr|) \tag{3.27}$$

Combining (3.26) and (3.27) we compute sum over all frequencies:

$$G_\sigma(t,x)=\frac{1}{2\pi}\sum(-1)^n\mathsf{e}^{-2\pi\mathrm{i}k_\sigma n}K_0(M|t+\mathrm{i}x+\mathrm{i}nr|) \tag{3.28}$$

3.3 Vacuum energy of free theory

It is well known that vacuum state of fermionic QFT is characterized by infinite number of quasiparticles occupying all states with negative energy. This configuration is sometimes referred to as Dirac sea. In 2 dimensions fermion is a 2-component spinor. If we parametrize the spatial part of momentum as

$$p=M\sinh\theta \tag{3.29}$$

then the vacuum energy is given by

$$E=-\sum_{n=-\infty}^{\infty}M\cosh\theta_n \tag{3.30}$$

The spectrum is defined by quasiperiodic boundary conditions:

$$p_nR=2\pi(n+\tfrac{1}{2}+k_\sigma) \tag{3.31}$$

Indeed,

$$\Psi_\sigma(x+R)=\mathsf{e}^{\mathrm{i}pR}\Psi_\sigma(x)=-\mathsf{e}^{2\pi\mathrm{i}k_\sigma}\Psi_\sigma(x) \tag{3.32}$$

so (3.31) can be inferred immediately. The energy of the Bukhovstov-Lipatov model in the absense of the interaction is given by a sum of contributions of different flavours:

$$E=E_+ + E_- \tag{3.33}$$

Then sum (3.30) reads

$$E_\sigma=-\sum_{n=-\infty}^{\infty}M\mathtt{E}_\sigma(n;r,M)\,,\qquad \mathtt{E}_\sigma(x;r,m)=\sqrt{\frac{m^2}{M^2}+\frac{4\pi^2(x+\frac{1}{2}+k_\sigma)^2}{r^2}} \tag{3.34}$$

where dimensionless parameter r appears such that $r=MR$. This sum is divergent (as Λ^2), and requires regularization. One way to do it is to use Pauli-Villars type regularization, computing difference between vacuum energies of theories with different masses.

$$E_\sigma^{(reg)}(R) = -\sum_{n=-\infty}^{\infty} M\left(\mathtt{E}_\sigma(n;r,M) - \mathtt{E}_\sigma(n;r,\mu)\right) \tag{3.35}$$

Parameter μ in (3.35) is a regulator mass which has nothing to do with mass in bosonic Lagrangian (2.42). Next step would be to subtract vacuum energy of theory on a plane. Latter is defined up to normalization, and we normalize it to be zero. Difference is normalization-independent. Introducing *regularized scaling function*

$$\mathfrak{f}_\sigma^{(reg)}(r) = \frac{R}{\pi}\left(E_\sigma^{(reg)}(R) - E_\sigma^{(reg)}(\infty)\right) \tag{3.36}$$

we can write

$$\mathfrak{f}_\sigma^{(reg)}(r) = -\sum_{n=-\infty}^{\infty}\frac{r}{\pi}\left(\mathtt{E}_\sigma(n;r,M) - \mathtt{E}_\sigma(n;r,\mu)\right) + \frac{r}{\pi}\int_{-\infty}^{\infty}\mathsf{d}x\left(\mathtt{E}_\sigma(x;r,M) - \mathtt{E}_\sigma(x;r,\mu)\right) \tag{3.37}$$

Quantity $E_\sigma^{(reg)}(\infty)$ appearing in (3.36) can be interpreted as specific bulk energy of free fermions in the limit $R \to \infty$. Indeed, formal change of integration variable leads to

$$E_\sigma^{(reg)}(\infty) = -\frac{M^2 R}{2\pi}\int_{-\infty}^{\infty}\mathsf{d}x\left(\sqrt{1+x^2} - \sqrt{\frac{\mu^2}{M^2}+x^2}\right) \tag{3.38}$$

This expression is proportional to R, i.e. energy density is constant in large R limit, so we can formally write

$$E(\infty) = \mathcal{E}R \tag{3.39}$$

The naive argument above does not take into account logarithmic divergence in (3.38) which, rigorously speaking, prevents us from change of integration variable. But for our purposes it is sufficient since combination (3.36) is convergent. Moreover, one can check that corrections to (3.38) which are not proportional to R vanish in large Λ limit. Using Poisson summation formula (3.15) we rewrite sum in (3.37) as

$$E_\sigma^{(reg)}(R) - E_\sigma^{(reg)}(\infty) = -\sum_{l\neq 0}\int_{-\infty}^{\infty}\mathsf{d}x\mathsf{e}^{2\pi \mathrm{i} l x}\left(\mathtt{E}_\sigma(x;r,M) - \mathtt{E}_\sigma(x;r,\mu)\right) \tag{3.40}$$

note that $l = 0$ contribution exactly cancels energy of theory on a plane. To compute integrals above let us differentiate w.r.t. r:

$$\frac{\mathsf{d}}{\mathsf{d}r}\left(E_\sigma^{(reg)}(R) - E_\sigma^{(reg)}(\infty)\right) = \tag{3.41}$$

$$= -\frac{2M}{r}\sum_{l\neq 0} D_\sigma\,|_{q=k}\int\limits_{-\infty}^{\infty}\mathsf{d}x\,\mathsf{e}^{2\pi \mathrm{i} qx}\left(\frac{1}{\mathsf{E}_\sigma(x;r,M)} - \frac{1}{\mathsf{E}_\sigma(x;r,\mu)}\right) \tag{3.42}$$

Differential operator D_σ in last expression is used to produce the nominator $\frac{4\pi^2}{r^2}(x + \frac{1}{2} + k_\sigma)^2$ from the exponent, so its expression reads

$$D_\sigma = -\frac{1}{r^2}\frac{\mathsf{d}^2}{\mathsf{d}q^2} - \frac{2\pi\mathrm{i}(2k_\sigma+1)}{r^2}\frac{\mathsf{d}}{\mathsf{d}q} + \frac{\pi^2(2k_\sigma+1)^2}{r^2} \tag{3.43}$$

Now we can split integrals for different masses without breaking convergence:

$$\int\limits_{-\infty}^{\infty}\frac{\mathsf{d}\,x\;\mathsf{e}^{2\pi \mathrm{i} lx}}{\mathsf{E}_\sigma(x;r,m)} = \frac{r}{2\pi}\int\limits_{-\infty}^{\infty}\mathsf{d}\theta\,\mathsf{e}^{\mathrm{i}lrmM^{-1}\sinh\theta - \pi \mathrm{i} l(1+2k_\sigma)} = \frac{r}{2\pi}(-1)^l\mathsf{e}^{-2\pi\mathrm{i}lk_\sigma}K_0(m|l|r/M) \tag{3.44}$$

Evaluating action of D on this answer we obtain

$$\begin{aligned}\frac{\mathsf{d}}{\mathsf{d}r}\left(E_\sigma^{(reg)}(R) - E_\sigma^{(reg)}(\infty)\right) &= \frac{M}{\pi}\sum_{l\neq 0}(-1)^{k_\sigma}\mathsf{e}^{-2\pi\mathrm{i}lk_\sigma}K_1'(|l|r)\\ &- \frac{M}{\pi}\sum_{l\neq 0}(-1)^{k_\sigma}\mathsf{e}^{-2\pi\mathrm{i}lk_\sigma}K_1'(|l|r\,\mu/M)\end{aligned} \tag{3.45}$$

It is not difficult to perform integration in r

$$E_\sigma^{(reg)}(R) - E_\sigma^{(reg)}(\infty) = \frac{M}{\pi}\sum_{l\neq 0}(-1)^l\mathsf{e}^{-2\pi\mathrm{i}lk_\sigma}\frac{1}{|l|}\left(K_1(lr) - \frac{M}{\mu}K_1(\mu\,lr/M)\right) \tag{3.46}$$

Now we can send regulator mass μ to infinity, and obtain

$$E_\sigma(R) - E_\sigma(\infty) = \frac{M}{\pi}\left(\sum_{l=1}^{\infty}(-1)^l\mathsf{e}^{-2\pi\mathrm{i}lk_\sigma}\frac{1}{l}K_1(lr) + \sum_{l=1}^{\infty}(-1)^l\mathsf{e}^{2\pi\mathrm{i}lk_\sigma}\frac{1}{k}K_1(lr)\right) \tag{3.47}$$

The last step in this derivation is to use integral representation for Macdonald function

$$K_s(z) = \frac{1}{2}\int\limits_{-\infty}^{\infty}\mathsf{e}^{s\theta - z\cosh\theta}\,\mathsf{d}\,\theta \tag{3.48}$$

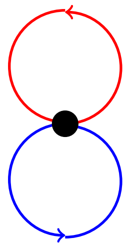


Figure 3.1: The diagram representing the first order perturbative contribution $\mathfrak{F}_1$ to the scaling function $\mathfrak{F}$. Red and blue lines represent propagators of Ψ_+ and Ψ_- respectively.

and sum up the Taylor series of logarithm

$$\begin{aligned}\mathfrak{f}_\sigma(r) &= -\frac{r}{2\pi^2}\int\limits_{-\infty}^{\infty} \mathsf{d}\,\theta\,\mathsf{e}^{\theta}\sum_{l=1}^{\infty}\frac{1}{l}\left(\mathsf{e}^{-lr\cosh\theta-2\pi\mathrm{i}lk_\sigma}+\mathsf{e}^{-lr\cosh\theta+2\pi\mathrm{i}lk_\sigma}\right)\\ &= -\frac{r}{2\pi^2}\int\limits_{-\infty}^{\infty}\mathsf{d}\,\theta\cosh\theta\log\left[\left(1+\mathsf{e}^{-r\cosh\theta+2\pi\mathrm{i}k_\sigma}\right)\left(1+\mathsf{e}^{-r\cosh\theta-2\pi\mathrm{i}k_\sigma}\right)\right]\end{aligned} \tag{3.49}$$

Thus we evaluated dimensionless scaling function

$$\begin{aligned}\mathfrak{f}_\sigma(r) &= \frac{R}{\pi}\big(E_\sigma(R)-\mathcal{E}_\sigma R\big)\\ &= -\frac{r}{2\pi^2}\int\limits_{-\infty}^{\infty}\mathsf{d}\,\theta\cosh\theta\log\left[\left(1+\mathsf{e}^{-r\cosh\theta+2\pi\mathrm{i}k_\sigma}\right)\left(1+\mathsf{e}^{-r\cosh\theta-2\pi\mathrm{i}k_\sigma}\right)\right]\end{aligned} \tag{3.50}$$

Full scaling function of Bukhvostov-Lipatov model in this approximation can be found as

$$\mathfrak{F}_0(r) = \frac{R}{\pi}\big(E(R)-\mathcal{E}R\big) = \mathfrak{f}_+(r)+\mathfrak{f}_-(r) \tag{3.51}$$

3.4 First order correction to the vacuum energy

The only diagram contributing to first order correction is shown on Fig.3.1. Then first order correction reads

$$\mathfrak{F}_1 = R^2\,\mathrm{Tr}(\gamma^\alpha\mathbf{S}_+(0))\mathrm{Tr}(\gamma_\alpha\mathbf{S}_-(0)) \tag{3.52}$$

Evaluating traces we shall get

$$\mathfrak{F}_1 = R^2\left(\partial_0 G_+(0)\partial_0 G_-(0)+\partial_1 G_+(0)\partial_1 G_-(0)\right) \tag{3.53}$$

In the limit $x \to 0, t \to 0$ derivative

$$\partial_0 K_0(M|t + ix + inr|) = MK_1(nr)\frac{t}{\sqrt{t^2 + (x + nr)^2}} \tag{3.54}$$

equals zero unless $n = 0$. In latter case the answer depends on the order in which limits $t \to 0$ and $x \to 0$ are taken, but is independent of r. Spatial derivative

$$\partial_1 K_0(M|t + ix + inr|) = MK_1(nr)\frac{x + nr}{\sqrt{t^2 + (x + nr)^2}} \tag{3.55}$$

tends to $MK_1(nr)$ for $n \neq 0$. To avoid the ambiguity related to terms with $n = 0$ we shall compute the same quantity in momentum space using different regularization. Loop momenta are quantised as

$$p_{1,+} = \frac{2\pi(n + \frac{1}{2} + k_+)}{R} \qquad p_{1,-} = \frac{2\pi(l + \frac{1}{2} + k_-)}{R} \tag{3.56}$$

In the momentum representation first order correction looks like

$$\begin{aligned}\mathfrak{F}_1 = &-\int\limits_{-\infty}^{\infty} \mathsf{d}\omega_1 \int\limits_{-\infty}^{\infty} \mathsf{d}\omega_2 \sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \mathrm{Tr}\left(\gamma^\alpha \frac{\hat{p}(\omega_1, n) + \mathrm{i}M}{p^2(\omega_1, n) + M^2} - \gamma^\alpha \frac{\hat{p}(\omega_1, n) + \mathrm{i}\mu}{p^2(\omega_1, n) + \mu^2}\right) \\ &\times \mathrm{Tr}\left(\gamma_\alpha \frac{\hat{p}(\omega_2, l) + \mathrm{i}M}{p^2(\omega_2, l) + M^2} - \gamma^\alpha \frac{\hat{p}(\omega_2, l) + \mathrm{i}\mu}{p^2(\omega_2, l) + \mu^2}\right)\end{aligned} \tag{3.57}$$

where we regularised propagators in the loops by subtracting similar propagators of fermion with mass μ, which is supposed to be extremely large and should be set to infinity in order to get physical answer. Introducing shortcut notation for regularized propagator

$$\hat{G}_{reg}(\omega, n, k; \mu) = \frac{1}{\omega^2 + \frac{4\pi^2(n+\frac{1}{2}+k)^2}{R^2} + M^2} - \frac{1}{\omega^2 + \frac{4\pi^2(n+\frac{1}{2}+k)^2}{R^2} + \mu^2} \tag{3.58}$$

we write the first-order correction as

$$\begin{aligned}\mathfrak{F}_1 = &-\int\limits_{-\infty}^{\infty} \mathsf{d}\omega_1 \int\limits_{-\infty}^{\infty} \mathsf{d}\omega_2 \sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \Big[\omega_1\omega_2 \hat{G}_{reg}(\omega_1, n, k_+; \mu)\, \hat{G}_{reg}(\omega_2, l, k_-; \mu) \\ &+ \frac{4\pi^2(n + \frac{1}{2} + k_+)(l + \frac{1}{2} + k_-)}{R^2} \hat{G}_{reg}(\omega_1, n, k_+; \mu)\, \hat{G}_{reg}(\omega_2, l, k_-; \mu)\Big]\end{aligned} \tag{3.59}$$

Let us integrate in ω_1,ω_2. First term in square brackets integrates to zero. Second is the product of two terms of the form

$$\mathfrak{F}_1 = -\sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\int \mathsf{d}\omega_1\mathsf{d}\omega_2 \frac{\pi^2(n+\frac{1}{2}+k_+)}{R}\mathcal{G}_{reg}(n,k_+;\mu)\frac{\pi^2(l+\frac{1}{2}+k_-)}{R}\mathcal{G}_{reg}(l,k_-;\mu) \tag{3.60}$$

where $\mathcal{G}_{reg}$ is a shortcut notation for

$$\mathcal{G}_{reg}(n,k;\mu) = \left(\frac{1}{\sqrt{\frac{4\pi^2(n+\frac{1}{2}+k)^2}{R^2}+M^2}} - \frac{1}{\sqrt{\frac{4\pi^2(n+\frac{1}{2}+k)^2}{R^2}+\mu^2}}\right) \tag{3.61}$$

Sum can be computed using Poisson summation formula (3.15)

$$\sum_{n=-\infty}^{\infty}\frac{\pi^2(n+\frac{1}{2}+k)}{R}\mathcal{G}_{reg}(n,k;\mu) = \sum_{l=-\infty}^{\infty}\int_{-\infty}^{\infty}\mathsf{d}y\mathsf{e}^{2\pi i l y}\frac{\pi(y+\frac{1}{2}+k)}{R}\mathcal{G}_{reg}(y,k;\mu) \tag{3.62}$$

The sum in right hand side can be evaluated using same trick as for free energy:

$$\sum_{l=-\infty}^{\infty}\int_{-\infty}^{\infty}\mathsf{d}y\mathsf{e}^{2\pi i l y}\frac{2\pi(y+\frac{1}{2}+k)}{r}\mathcal{G}_{reg}(y,k;\mu) = \sum_{k=-\infty}^{\infty} D\,|_{q=l}\int_{-\infty}^{\infty}\mathsf{d}y\mathsf{e}^{2\pi \mathrm{i} q y}\mathcal{G}_{reg}(y,k;\mu) \tag{3.63}$$

In this case differential operator D is given by

$$D = -\frac{\mathrm{i}}{r}\frac{\mathsf{d}}{\mathsf{d}q} + \frac{\pi(2k+1)}{r} \tag{3.64}$$

We have already evaluated integral above: for nonzero q it reads

$$\int_{-\infty}^{\infty}\mathsf{d}\,y\;\mathsf{e}^{2\pi \mathrm{i} q y}\,\mathcal{G}_{reg}(y,k;\mu) = \frac{r}{2\pi}\mathsf{e}^{-\pi \mathrm{i} q(2k_+ +1)}\left(K_0(|q|r) - K_0\left(|q|\mu r/M\right)\right) \tag{3.65}$$

Macdonald function K_0 at small values of argument admits the following expansion:

$$K_0(z) = -\log\frac{z}{2} + \psi(1) + O(z^2) \tag{3.66}$$

Then for $q \ll 1$

$$K_0(qr) - K_0\left(\mu q r/M\right) = \log\left(\frac{\mu}{M}\right) + O(q^2) \tag{3.67}$$

and term $l = 0$ disappears under action of operator D. Thus one flavour loop yields

$$\langle \Psi_+^\dagger(0)\Psi_+(0)\rangle = -\frac{r}{\pi}\sum_{l=1}^{\infty}(-1)^l \sin(2\pi l k_+)\left(K_1(lr) - \mu M^{-1}K_1(\mu M^{-1}lr)\right) \tag{3.68}$$

and full first order correction to vacuum energy

$$\begin{aligned}\mathfrak{F}_1 = &-\frac{r^2}{\pi^2}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}(-1)^l(-1)^n \sin(2\pi n k_+)\sin(2\pi l k_-)\\ &\times \left(K_1(nr) - \mu M^{-1}K_1(\mu M^{-1}nr)\right)\left(K_1(lr) - \mu M^{-1}K_1(\mu M^{-1}lr)\right)\end{aligned} \tag{3.69}$$

Now limit $\mu \to \infty$ can be safely taken

$$\mathfrak{F}_1 = -\frac{r^2}{\pi^2}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}(-1)^l(-1)^n \sin(2\pi n k_+)\sin(2\pi l k_-)K_1(nr)K_1(lr) \tag{3.70}$$

From (3.70) and (3.50) one can easily verify that

$$\mathfrak{F}_1 = -4\mathfrak{q}(r,k_+)\mathfrak{q}(r,k_-)\,, \qquad \mathfrak{q}(r,k) = \frac{1}{4}\frac{\partial \mathfrak{f}(r,k)}{\partial k} \tag{3.71}$$

3.5 Second order correction to the vacuum energy

The type I diagram gives the contribution

$$\mathfrak{F}_2^{(I)} = -\frac{\pi}{2}\, r^2 \int_{D_\epsilon} \mathsf{d}^2x\ \mathrm{Tr}\big(\boldsymbol{S}_+(-\mathbf{x})\,\gamma_a\,\boldsymbol{S}_+(\mathbf{x})\,\gamma_b\big)\ \mathrm{Tr}\big(\boldsymbol{S}_-(-\mathbf{x})\,\gamma^a\,\boldsymbol{S}_-(\mathbf{x})\,\gamma^b\big)\ . \tag{3.72}$$

Because of the UV divergence at $\mathbf{x} = 0$, the integration domain D_ϵ here is chosen to be the cylinder

$$D = \{\mathbf{x} = (x_0, x_1)\mid -\infty < x_0 < \infty, x_1 \equiv x_1 + R\} \tag{3.73}$$

without an infinitesimal hole $|\mathbf{x}| < \epsilon$. One can show that, as ϵ tends to zero,

$$\mathfrak{F}_2^{(I)} = -\Big(\frac{R}{2\pi\epsilon}\Big)^2 + \sum_{\sigma=\pm}\Big(\mathfrak{t}(r,k_\sigma) - \frac{r}{2\pi}\,\log\big(\tfrac{M\epsilon}{2}\,\mathsf{e}^{\gamma_E-\frac{1}{2}}\big)\Big)^2 + \text{finite}\ , \tag{3.74}$$

where

$$\mathfrak{t} = -\pi\,\frac{\partial \mathfrak{f}}{\partial r}\ . \tag{3.75}$$

Indeed, the trace evaluates to

$$\begin{aligned}\mathrm{Tr}\big(\boldsymbol{S}_+(-\mathbf{x})\,\gamma_a\,\boldsymbol{S}_+(\mathbf{x})\,\gamma_b\big)\ \mathrm{Tr}\big(\boldsymbol{S}_-(-\mathbf{x})\,\gamma^a\,\boldsymbol{S}_-(\mathbf{x})\,\gamma^b\big) =& \\ & 4M^4G_+(x)G_+(-x)G_-(x)G_-(-x) \\ & +4\sum_\sigma \partial_aG_+(x)\partial_aG_-(\sigma x)\partial_bG_+(-x)\partial_bG_-(-\sigma x) \\ & -4\partial_aG_+(x)\partial_aG_+(-x)\partial_bG_-(x)\partial_bG_-(-x)\end{aligned} \tag{3.76}$$

To find dependence on cutoff we expand this at $|\mathbf{x}|=\epsilon$. We need to keep terms that diverge at least as ϵ^{-2}.

$$\begin{aligned}&\partial_aG_\sigma(\mathbf{x})\partial^aG_{\sigma'}(-\sigma''\mathbf{x}) = \frac{1}{\epsilon^2} - M^2\log(M\epsilon) - \frac{1}{2}\left(-1+2\gamma_E-2\log 2\right)M^2 \\ &+\frac{2\mathrm{i}M}{\epsilon}\sum_{n=1}^{\infty}(-1)^n\left(\sin(2\pi k_\sigma n)-\sigma''\sin(2\pi k_{\sigma'}n)\right)K_1(nr)\sin\phi \\ &+\sum_{n=1}^{\infty}2(-1)^n\left(\cos(2\pi k_\sigma)+\cos(2\pi k_{\sigma'})\right)\left(M^2K_1'(nr)\sin^2\phi+K_1(nr)\frac{M^2\cos^2\phi}{nr}\right) \\ &+4M^2\sigma\sum_{n,m=1}^{\infty}\sin(2\pi k_\sigma n)\sin(2\pi k_{\sigma'}m)K_1(nr)K_1(mr)+O(\varepsilon)\end{aligned} \tag{3.77}$$

Substituting this into (3.76) and integrating over ϵ we get (3.74). In fact, since $E_{\mathbf{k}} = R\,\mathcal{E}+\frac{\pi}{R}\,\mathfrak{F}$, the quadratic divergence $\propto 1/\epsilon^2$ should be relocated to the specific bulk energy.

The type II diagrams from Fig. 3.2 leads to the UV finite integral over the whole cylinder D:

$$\mathfrak{F}_2^{(II)} = \frac{\pi}{2}\,r^2\int_D \mathsf{d}^2x\sum_{\sigma=\pm}\mathrm{Tr}\big(\boldsymbol{S}_\sigma(0)\,\gamma_a\big)\,\mathrm{Tr}\big(\boldsymbol{S}_{-\sigma}(-\mathbf{x})\,\gamma^a\,\boldsymbol{S}_{-\sigma}(\mathbf{x})\,\gamma^b\big)\,\mathrm{Tr}\big(\boldsymbol{S}_\sigma(0)\,\gamma_b\big) \tag{3.78}$$

Finally, the counterterm $\propto g_1$ in (3.4) contributes through the type III diagrams, schematically visualized in Fig. 3.2. This can be written in the form $\frac{2g_1}{\pi}\,\mathfrak{F}_2^{(III)}$ with

$$\begin{aligned}\mathfrak{F}_2^{(III)} &= \frac{1}{4}\,R^2\sum_{\sigma=\pm}\Big(\mathrm{Tr}\big(\boldsymbol{S}_\sigma(0)\gamma^a\big)\,\mathrm{Tr}\big(\boldsymbol{S}_\sigma(0)\gamma_a\big)-\mathrm{Tr}\big(\boldsymbol{S}_\sigma(0)\gamma^a\boldsymbol{S}_\sigma(0)\gamma_a\big)\Big) \\ &= R^2\sum_{\sigma=\pm}\big(\langle\,\psi^\dagger_{\sigma,+}\,\psi_{\sigma,+}(0)\,\rangle\langle\,\psi^\dagger_{\sigma,-}\,\psi_{\sigma,-}(0)\,\rangle-\langle\,\psi_{\sigma,-}\,\psi^\dagger_{\sigma,+}(0)\,\rangle^2\big)\ .\end{aligned} \tag{3.79}$$

Contrary to the one point functions $\langle\,\psi^\dagger_{\sigma,+}\,\psi_{\sigma,+}(0)\,\rangle$ appearing in (3.52), the condensate $\langle\, \psi_{\sigma,-}\, \psi^{\dagger}_{\sigma,+}(0)\,\rangle$ diverges logarithmically:

$$\langle\, \psi_{\sigma,-}\, \psi^{\dagger}_{\sigma,+}(0)\,\rangle = \langle\, \psi_{\sigma,+}\, \psi^{\dagger}_{\sigma,-}(0)\,\rangle = \frac{1}{R}\, \mathfrak{t}(r,k_\sigma) - \frac{M}{2\pi}\, \log\Big(\frac{M\epsilon}{2}\, \mathsf{e}^{\gamma_E - \frac{1}{2} + C}\Big)\ , \tag{3.80}$$

where C is some constant. Indeed, the singular term originates from

$$G^2(\epsilon) = \left(\log\epsilon + \sum_{n=1}^{\infty} \cos(2\pi n k_\sigma) K_0(nr) + O(\epsilon)\right)^2 \tag{3.81}$$

Since

$$\mathfrak{F}_2^{(I)} + \mathfrak{F}_2^{(III)} + \Big(\frac{R}{2\pi\epsilon}\Big)^2 = -\frac{r^2}{\pi^2}\, C\ \log(M\epsilon) + \text{finite}\ , \tag{3.82}$$

the UV divergence $\propto \log^2(\epsilon)$ is canceled from the sum of types I and III diagrams if we choose $g_1 = \frac{g^2}{2\pi} + O(g^3)$. As well as the quadratic divergence, the remaining logarithmic divergence should be relocated to the specific bulk energy. Expanding $\cos^2\big(\frac{\pi\delta}{2}\big)$ in (3.11) one can find the value of the constant C:

$$C = \frac{\pi^2}{4}\ . \tag{3.83}$$

This way the second order correction takes the form

$$\mathfrak{F}_2 = \frac{r^2}{4\pi^2}\, \mathcal{C}_2 + \lim_{\epsilon\to 0}\Big[\sum_{\alpha=I,II,III} \mathfrak{F}_2^{(\alpha)} + \Big(\frac{R}{2\pi\epsilon}\Big)^2 + \frac{r^2}{4}\, \log\Big(\frac{M\epsilon}{2}\, \mathsf{e}^{\gamma_{\mathrm{E}} - \frac{1}{2}}\Big)\Big]\ , \tag{3.84}$$

where the finite constant should be adjusted to ensure that scaling function vanishes in large-r limit. It reads explicitly as

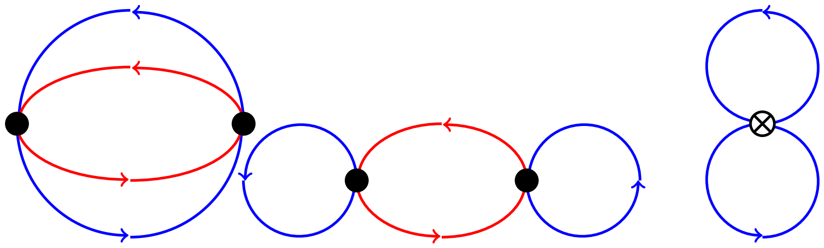
$$\mathcal{C}_2 = \frac{\pi^4}{8} - \frac{1}{2} - \frac{1}{4}\, \psi''\big(\tfrac{1}{2}\big)\ . \tag{3.85}$$

Using integral representation of Macdonald functions

$$K_s(z) = \frac{1}{2}\int_{-\infty}^{\infty} \mathsf{d}\theta\ \mathsf{e}^{s\theta - z\cosh(\theta)}\ , \tag{3.86}$$

to integrate over spatial variable in (3.72,3.78) and leaving only terms with $n = 0, \pm 1$ in expressions of propagators one can show that

$$\begin{aligned} \mathfrak{F}_2(r,\mathbf{k}) \;&=\; -\frac{1}{2}\,\big(1 + c(2k_1) + c(2k_2)\big)\ r^2\, K_0(2r) \qquad (3.87)\\ &-\;\big(1 - c(2k_1)c(2k_2)\big)\ r\int_{-\infty}^{\infty}\frac{\mathsf{d}\nu}{\pi}\,\frac{\nu^2\, K_{\mathrm{i}\nu}(r) K_{1+\mathrm{i}\nu}(r)}{\sinh^2(\frac{\pi\nu}{2})} + o\big(\mathsf{e}^{-2r}\big)\ , \end{aligned}$$



(a) $\mathfrak{F}_2^{(I)}$ (b) $\mathfrak{F}_2^{(II)}$ (c) $\mathfrak{F}_2^{(III)}$

Figure 3.2: Diagrams contributing to vacuum energy in second order in g. The contribution of counterterm is visualised with type III diagrams, where counterterm vertex should be understood as a sum over possible orientations of 4-fermionic vertex.

where shortcut notation $c(k) = \cos(\pi k)$ was used. Also in eq. (3.87) and below, the symbol $o\big(\mathrm{e}^{-2r}\big)$ denotes a remaining term that decays faster than $r^{-N}\,\mathrm{e}^{-2r}$ for any positive N as $r \to +\infty$. Notice that the normalization condition $\mathfrak{F}_2 = o\big(\mathrm{e}^{-r}\big)$ is in accordance with an absence of the finite renormalization of the fermion mass. It can be used for fixing the constant C in (3.80) and hence avoid any reference to the exact relation (3.11). It is straightforward to find large r asymptotics of one-loop correction (3.70) to scaling function:

$$\mathfrak{F}_1(r, \mathbf{k}) = \frac{2}{\pi^2}\,\big(c(2k_1) - c(2k_2)\big)\; r^2\, K_1^2(r) + o\big(\mathrm{e}^{-2r}\big)\ . \tag{3.88}$$

Thus, at least at the first-two perturbative orders, the leading large-r behavior of the scaling function $\mathfrak{F}$ is defined by $\mathfrak{F}_0$ only and therefore

$$\mathfrak{F}(r, \mathbf{k}) = -\frac{4}{\pi^2}\; c(k_1)\, c(k_2)\; r\, K_1(r) + o(\mathrm{e}^{-r})\ . \tag{3.89}$$

This can be understood as follows. The leading large-R behavior comes from the virtual fermions trajectories winding once around the Matsubara circle. Such trajectories should be counted with the phase factor $\mathrm{e}^{\mathrm{i}\pi(\sigma_1 k_1 + \sigma_2 k_2)}$ and, therefore, the summation over four possible sign combinations with $\sigma_{1,2} = \pm 1$ results in eq.(3.89).

Short distance expansion. Conformal Perturbation Theory

4.1 Conformal limit

In the limit $\mu \to 0$ bosonic formulation of Bukhvostov-Lipatov model

$$\mathcal{L}_{BL} = \frac{1}{16\pi}\left((\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2\right) + 4\mu\cos(\tfrac{1}{2}\sqrt{a_1}\phi_1)\cos(\tfrac{1}{2}\sqrt{a_2}\phi_2) \tag{4.1}$$

becomes a theory of two free massless bosonic fields [1]

$$\mathcal{L} = \frac{1}{16\pi}\left((\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2\right) \tag{4.2}$$

Quasiperiodic boundary conditions imposed on fermionic fields in Lagrangian (3.4) imply that states of bosonic conformal field theory are build on top of **k**-vacuum. It means that we are considering a subsector of bosonic CFT where all states have given eigenvalues of operators $\pi_{0,\pm}$. Indeed, from representation (2.48) of fermionic field we expect them to acquire phase

$$2\pi k_\pm = \langle : \int_0^R \dot{\chi}_\pm(x)\mathsf{d}x : \rangle \tag{4.3}$$

Free massless bosonic field operator can be decomposed as

$$\phi = \phi_0 - \frac{\mathrm{i}}{R}\pi_0 t + \frac{\mathrm{i}}{\sqrt{4\pi}}\sum_{k\neq 0}\frac{1}{k}\left(z^k a_k + \bar{z}^k \bar{a}_k\right) \quad z = \mathsf{e}^{\frac{2\pi(t+\mathrm{i}x)}{R}} \tag{4.4}$$

where operators $\phi_0, \pi_0, a, \bar{a}$ satisfy commutation relations

$$[a_n, a_m] = n\delta_{n+m,0}\,, \qquad [\bar{a}_n, \bar{a}_m] = n\delta_{n+m,0}\,, \qquad [a_n, \bar{a}_m] = 0 \tag{4.5}$$

[1] Note that in the book DiFrancesco et al. [81] different normalisation is used, i.e. $\mathcal{L}_{FB} = \frac{1}{8\pi}(\partial_\mu\phi)^2$

$$[\phi_0, a_n] = 0\,, \qquad [\phi_0, \bar{a}_n] = 0\,, \qquad [\phi_0, \pi_0] = \mathrm{i}\,, \qquad [\pi_0, a_n] = 0\,, \qquad [\pi_0, \bar{a}_n] = 0 \tag{4.6}$$

Operators $\pi_{0,i}$ in theory (4.2) commute with Hamiltonian, so the Fock space can be decomposed into sectors labeled by the pair of eigenvalues $P_\pm$. We can think of "charged" vacuum $|\mathbf{k}\rangle$ as a result of action of vertex operator

$$|\mathbf{k}\rangle =: \mathcal{V}_{\kappa_1}(0) :: \mathcal{V}_{\kappa_2} : |0\rangle \tag{4.7}$$

Unless otherwise specified all vacuum averages in bosonic theory on a cylinder throughout this chapter will be computed over $\mathbf{k}$-vacuum. Evaluation of integral in r.h.s. of (4.3) in conformal field theory leads to

$$k_\pm = P_\pm \tag{4.8}$$

As before, we are interested in scaling function of Bukhvostov-Lipatov model.

$$\mathfrak{F}(r, \mathbf{k}) = \frac{R}{\pi}\left(E_{\mathbf{k}} - R\,\mathcal{E}\right)\,. \tag{4.9}$$

In leading approximation (unperturbed conformal field theory) it is proportional to the central charge of the theory

$$c_{\mathbf{k}} = -6\,\mathfrak{F}(0, \mathbf{k}) \tag{4.10}$$

which for representation of Virasoro algebra with $\mathbf{k}$-vacuum as its highest weight vector equals

$$c_{\mathbf{k}} = \sum_{i=1,2}\left(1 - 6a_i k_i^2\right) \tag{4.11}$$

4.2 Vertex operators in CFT

Recall the notion of vertex operator in 2-dimensional conformal field theory of a free scalar boson [81, 82].

$$\mathcal{V}_\alpha(z, \bar{z}) =: \mathsf{e}^{\mathrm{i}\alpha\phi(z,\bar{z})} : \tag{4.12}$$

Vertex operator can be decomposed into holomorphic and antiholomorphic parts

$$\mathcal{V}_\alpha(z, \bar{z}) = V_\alpha(z) \otimes \bar{V}_\alpha(\bar{z}) \tag{4.13}$$

Vertex operators are conformal primaries, i.e. they transform under conformal transformations as

$$\tilde{\mathcal{V}}_\alpha(w, \bar{w}) = \left(\frac{dw}{dz}\right)^{-h_\alpha}\left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}_\alpha}\mathcal{V}_\alpha(z, \bar{z}) \tag{4.14}$$

with conformal dimensions h_α,$\bar{h}_\alpha$ given by

$$h_\alpha = \bar{h}_\alpha = \alpha^2 \tag{4.15}$$

The correlators of vertex operators are known explicitly. On a plane, they read [81, 82].

$$\langle \mathcal{V}_{\alpha_1}(z_1,\bar{z}_1)\mathcal{V}_{\alpha_2}(z_2,\bar{z}_2)...\mathcal{V}_{\alpha_n}(z_n,\bar{z}_n)\rangle_{plane} = \prod_{i>j} |z_i - z_j|^{4\alpha_i\alpha_j} \tag{4.16}$$

if neutrality condition

$$\sum_{i=1}^{n} \alpha_i = 0 \tag{4.17}$$

is satisfied, and are equal to zero otherwise. To compute their correlators on a cylinder in charged sector, we make use of (4.13) and the fact that **k**-vacuum is related to neutral one as

$$|\mathbf{k}\rangle = \mathcal{V}^{(1)}_{\kappa_1}(0)\mathcal{V}^{(2)}_{\kappa_2}(0)|0\rangle\,, \quad \langle\mathbf{k}| = \langle 0|\mathcal{V}^{(2)}_{-\kappa_2}(\infty)\mathcal{V}^{(1)}_{-\kappa_1}(\infty) \tag{4.18}$$

with

$$\mathcal{V}^{(i)}_{\alpha} =: \mathsf{e}^{\mathrm{i}\alpha\phi_i} :, \qquad \kappa_i = \tfrac{1}{2}\sqrt{a_i}k_i\,, \qquad k_1 = k_+ + k_-\,, \qquad k_2 = k_+ - k_- \tag{4.19}$$

The notation $\mathcal{V}(\infty)$ in (4.18) should be understood as a limit

$$\mathcal{V}_{-\alpha}(\infty) = \lim_{R\to\infty} R^{4\alpha^2}\,\mathcal{V}_{-\alpha}(R) \tag{4.20}$$

to ensure proper normalization

$$\langle\, \mathbf{k} \mid \mathbf{k}\,\rangle = 1 \tag{4.21}$$

Thus we obtain for the correlator of vertex operators on a cylinder

$$\langle \mathcal{V}^{(i)}_{\alpha_1}(\mathbf{x}_1)\mathcal{V}^{(i)}_{\alpha_2}(\mathbf{x}_2)...\mathcal{V}^{(i)}_{\alpha_n}(\mathbf{x}_n)\rangle = \left(\frac{2\pi}{R}\right)^{2\sum_{j=1}^{n}\alpha_j^2} \prod_{j=1}^{n} |z_j|^{2\alpha_j^2 - 4\alpha_j\kappa_i} \prod_{j=1}^{n-1}\prod_{l=j+1}^{n} |z_l - z_j|^{4\alpha_l\alpha_j} \tag{4.22}$$

with

$$\mathbf{x}_j = (t_j, x_j)\,, \qquad z_j = \mathsf{e}^{\frac{2\pi(t_j + \mathrm{i}x_j)}{R}} \tag{4.23}$$

4.3 Conformal Perturbation Theory

Now, following works of Zamolodchikov [83, 84], Bazhanov et al. [85] etc we are ready to formulate conformal perturbation theory for Bukhvostov-Lipatov model.

We consider our theory as a 2-dimensional conformal perturbation theory of 2 free scalar bosons perturbed by a sum of 4 vertex operators

$$\mathcal{L}_{BL} = \mathcal{L}_{1,FB} + \mathcal{L}_{2,FB} + \mu \left(\sum_{\sigma,\sigma'=\pm} \mathcal{V}_{\sigma\alpha_1} \mathcal{V}_{\sigma'\alpha_2} \right) \tag{4.24}$$

where we have introduced the notations

$$\alpha_1 = \frac{\sqrt{a_1}}{2} \qquad \alpha_2 = \frac{\sqrt{a_2}}{2} \tag{4.25}$$

Then the vacuum energy is given by

$$ER = \lim_{T\to\infty} \frac{R}{T} \log Z = E_0 R + \sum_{n=1}^{\infty} \mu^{2n} \lim_{T\to\infty} \frac{R}{T} \int \prod_{j=1}^{2n} \mathsf{d}^2 x_j \langle \prod_{k=1}^{2n} U(x_k) \rangle_{cyl,con} \tag{4.26}$$

where

$$U(x) = \sum_{\sigma,\sigma'=\pm} \mathcal{V}_{\sigma\alpha_1}(x) \mathcal{V}_{\sigma'\alpha_2} \tag{4.27}$$

The fact that sum is composed of even powers of μ only is a consequence of neutrality condition (4.17), as it will be evident shortly. Individual terms of this expansion read

$$\begin{aligned}
&\mu^{2n} \lim_{T\to\infty} \frac{R}{T} \int \prod_{j=1}^{2n} \mathsf{d}^2 \mathbf{x}_j \, \langle \prod_{j=1}^{2n} \mathcal{V}^{(1)}_{\sigma_j\alpha_1}(\mathbf{x}_j) \rangle \, \langle \prod_{j=1}^{2n} \mathcal{V}^{(2)}_{\sigma'_j\alpha_2}(\mathbf{x}_j) \rangle_{conn} = \\
&\mu^{2n} \left(\frac{R}{2\pi} \right)^{n(a_1+a_2)+2-4n} \int \mathsf{d}^2 z_2 \ldots \mathsf{d}^2 z_n \prod_{j=2}^{2n} \frac{|z_j|^{-1+\nu+\sigma_j a_1 k_1 + \sigma'_j a_2 k_2}}{|z_j - 1|^{-a_1\sigma_1\sigma_j - a_2\sigma'_1\sigma'_j}} \\
&\times \prod_{i=j+1}^{2n} |z_j - z_i|^{a_1\sigma_i\sigma_j + a_2\sigma'_i\sigma'_j} - \text{contractions}
\end{aligned} \tag{4.28}$$

Introducing parameter ν such that

$$a_1 + a_2 = 2 + 2\nu \tag{4.29}$$

we can write down CPT series as

$$E_{\mathbf{k}} = \frac{\pi}{R} \sum_{n=0}^{\infty} e_n^{(\nu)} \lambda^{2n}, \qquad \lambda = 2\pi\mu \left(\frac{2\pi}{R} \right)^{\nu-1} \tag{4.30}$$

where coefficients e_n and expansion parameter λ are dimensionless. We are interested in case $\nu = 0$, but for the sake of regularization we shall keep it finite during the calculation.

4.4 First order correction

The first non-zero correction to CFT value of scaling function reads

$$e_1 = I(p_+) + I(p_-)\,, \qquad p_\pm = \tfrac{1}{2}a_1 k_1 \pm \tfrac{1}{2}a_2 k_2 \tag{4.31}$$

where

$$I(p_\sigma) = \int \frac{d^2\mathbf{x}}{R^2} \langle \mathcal{V}_{-\alpha_1}(0)\mathcal{V}_{\alpha_1}(x)\rangle\langle \mathcal{V}_{-\sigma\alpha_2}(0)\mathcal{V}_{\sigma\alpha_2}(x)\rangle \tag{4.32}$$

Evaluating that, one obtains

$$I(p) = \left(\frac{2\pi}{R}\right)^{\frac{a_1+a_2}{2}-1} \int \frac{\mathsf{d}^2 z}{|z|} \frac{|z|^{-(a_1k_1+a_2k_2)}}{|\sqrt{z} - \frac{1}{\sqrt{z}}|^{a_1+a_2}} \tag{4.33}$$

We're interested in computing the following integral

$$I(p) = \frac{1}{R^2}\int\limits_{-\infty}^{\infty} \mathsf{d}t \int\limits_{0}^{R} \mathsf{d}x \frac{4^{-\nu}\pi \mathsf{e}^{-\frac{2\pi}{R}(\nu+2p)t}}{\left(\sinh(\frac{\pi}{R}(t+\mathrm{i}x))\sinh(\frac{\pi}{R}(t-\mathrm{i}x))\right)^{1+\nu}} \tag{4.34}$$

Let us perform integration in x. Relevant part can be rewritten as

$$\int\limits_0^R \frac{\mathsf{d}x}{\left(\frac{1}{2}(\cosh\frac{2\pi t}{R} - \cos\frac{2\pi x}{R})\right)^{1+\nu}} = \frac{2^{1+\nu}}{\left(\cosh\frac{2\pi t}{R}\right)^{1+\nu}} \int\limits_0^R \frac{\mathsf{d}x}{\left(1 - \frac{\cos\frac{2\pi x}{R}}{\cosh\frac{2\pi t}{R}}\right)^{1+\nu}} \tag{4.35}$$

Let us assume that $t \neq 0$. Then the integrand can be expanded as convergent Taylor series

$$\frac{1}{\left(1 - \frac{\cos\frac{2\pi x}{R}}{\cosh\frac{2\pi t}{R}}\right)^{1+\nu}} = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{\Gamma(\nu+k+1)}{\Gamma(\nu+1)} \left(\frac{\cos\frac{2\pi x}{R}}{\cosh\frac{2\pi t}{R}}\right)^k \tag{4.36}$$

Each term can be easily integrated in x:

$$\int\limits_0^R \cos\left(\frac{2\pi x}{R}\right)^{2k} \mathsf{d}x = R\frac{(2k)!}{(k!)^2}2^{-2k} \tag{4.37}$$

For odd powers integral equals to zero. Then integral in x reads

$$\int\limits_0^R \frac{dx}{\left(1 - \frac{\cos\frac{2\pi x}{R}}{\cosh\frac{2\pi t}{R}}\right)^{1+\nu}} = R\sum_{k=0}^{\infty} \frac{1}{(k!)^2}\frac{\Gamma(\nu+2k+1)}{\Gamma(\nu+1)}\frac{1}{\left(2\cosh\frac{2\pi t}{R}\right)^{2k}} \tag{4.38}$$

This series is convergent even for $t = 0$, so this case does not require special treatment. It may be considered as an analytical continuation from region $\nu < -\frac{1}{2}$, where integral in x converges for arbitrary t. For Bukhvostov-Lipatov model we're interested in expansion near $\nu = 0$ (Integral diverges and requires regularisation there). Next step will be to integrate in t.

$$4\pi R \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \frac{\Gamma(\nu + 2k + 1)}{\Gamma(\nu + 1)} \int_{-\infty}^{\infty} \mathrm{d}t \frac{\mathrm{e}^{-\frac{2\pi}{R}(\nu+2p)t}}{\left(2\cosh \frac{2\pi}{R} t\right)^{\nu+2k+1}} \tag{4.39}$$

Introducing $\tau = \mathrm{e}^{\frac{2\pi t}{R}}$ we rewrite integral as

$$\int_0^{\infty} \mathrm{d}\tau \frac{\tau^{2k-2p}}{\left(\tau^2 + 1\right)^{\nu+2k+1}} = \int_0^{\infty} \mathrm{d}\tau \frac{\tau^{k-p-\frac{1}{2}}}{2(\tau + 1)^{\nu+2k+1}} = \tfrac{1}{2} B(k - p + \tfrac{1}{2}, \nu + k + p + \tfrac{1}{2}) \tag{4.40}$$

where we used integral representation of Euler beta function. Finally we need to sum the series:

$$\sum_{k=0}^{\infty} \frac{1}{k!} \frac{\Gamma(k - p + \frac{1}{2})\Gamma(\nu + k + p + \frac{1}{2})}{\Gamma(1 + \nu) k!} = \frac{\Gamma(\frac{1}{2} - p)\Gamma(\nu + p + \frac{1}{2})\Gamma(-\nu)}{\Gamma(p + \frac{1}{2})\Gamma(\frac{1}{2} - \nu - p)\Gamma(1 + \nu)} \tag{4.41}$$

where we recognized hypergeometric series and took advantage of the fact that

$${}_2F_1(\tfrac{1}{2} - p, \nu + p + \tfrac{1}{2}, 1; 1) = \frac{\Gamma(1)\Gamma(-\nu)}{\Gamma(\frac{1}{2} + p)\Gamma(\frac{1}{2} - p - \nu)} \tag{4.42}$$

Bukhvostov-Lipatov model with chosen renormalization scheme corresponds to the case $\nu = 0$. Expression (4.41) diverges as ν^{-1}:

$$I(p) = -\frac{1}{\nu} - 2\gamma_E - \psi(\tfrac{1}{2} + p) - \psi(\tfrac{1}{2} - p) \tag{4.43}$$

On the other the difference

$$I(p) - I(0) = 2\psi(\tfrac{1}{2}) - \psi(\tfrac{1}{2} + p) - \psi(\tfrac{1}{2} - p) \tag{4.44}$$

is finite. Instead of considering nonzero ν we can exclude small disk of radius ε from the domain of integration. Integral (4.34) diverges as $\log \varepsilon$. Moreover, it is straightforward to compute $I(0)$ in this regularisation:

$$I(0) = 2 \log \left(\frac{2\pi\varepsilon}{R} \right) \tag{4.45}$$

Thus we can write down the expansion of vacuum energy in this form:

$$\frac{RE_{\mathbf{k}}}{\pi} \asymp -\frac{1}{3}+\frac{4p_1^2}{1-\delta}+\frac{4p_2^2}{1+\delta}-(\mu R)^2\,\Big(e_1(0)-4\,\log\big(\tfrac{2\pi}{R}\,\epsilon\,\mathsf{e}^{\gamma_E-\frac{1}{2}}\big)\Big)-\sum_{n=2}^{\infty}e_n(\delta)\,(\mu R)^{2n}\;, \tag{4.46}$$

where explicitly

$$\begin{aligned} e_1(0) = -2 &- \psi\big(\tfrac{1}{2}+p_1+p_2\big) - \psi\big(\tfrac{1}{2}-p_1-p_2\big) \\ &- \psi\big(\tfrac{1}{2}+p_1-p_2\big) - \psi\big(\tfrac{1}{2}-p_1+p_2\big) \end{aligned} \tag{4.47}$$

and

$$p_1 = \frac{1}{2}\,(1-\delta)\,k_1\;, \qquad p_2 = \frac{1}{2}\,(1+\delta)\,k_2\;. \tag{4.48}$$

According to Fateev [73], parameters μ and M in chosen regularization scheme should be related as

$$\mu = \tfrac{M}{2\pi}\,\cos\big(\tfrac{\pi\delta}{2}\big)\;. \tag{4.49}$$

Expansion of extensive part of energy can be written as follows:

$$\mathcal{E} = \pi\mu^2\,\Big(4\,\log\big(\pi\mu\epsilon\,\mathsf{e}^{\gamma_E-\frac{1}{2}}\big)+\psi\big(\tfrac{1+\delta}{2}\big)+\psi\big(\tfrac{1-\delta}{2}\big)-2\,\psi\big(\tfrac{1}{2}\big)\Big)\;. \tag{4.50}$$

Therefore scaling function can be represented as asymptotic power series in ρ:

$$\mathfrak{F}(r,\mathbf{k}) \asymp -\frac{1}{3}+2k_+^2+2k_-^2-4\delta\,k_+k_- -16\,\rho^2\,\log(\rho)-\sum_{n=1}^{\infty}e_n(\delta)\,(2\rho)^{2n}\;, \tag{4.51}$$

where $k_\pm = \frac{1}{2}(k_1 \pm k_2)$, $\rho = \frac{r}{4\pi}\,\cos\big(\frac{\pi\delta}{2}\big)$ and

$$e_1(\delta) = e_1(0)+\psi\big(\tfrac{1+\delta}{2}\big)+\psi\big(\tfrac{1-\delta}{2}\big)-2\,\psi\big(\tfrac{1}{2}\big)\;. \tag{4.52}$$

4.5 Second order correction

Now the fact that correlators in question must be connected ones starts to affect the calculations. Connected 4-point correlator reads (assuming 1- and 3-point functions are zero)

$$\begin{aligned} &\langle U(\mathbf{x}_1)U(\mathbf{x}_2)U(\mathbf{x}_3)U(\mathbf{x}_4)\rangle_{con} = \langle U(\mathbf{x}_1)U(\mathbf{x}_2)U(\mathbf{x}_3)U(\mathbf{x}_4)\rangle \\ &-\langle U(\mathbf{x}_1)U(\mathbf{x}_2)\rangle\langle U(\mathbf{x}_3)U(\mathbf{x}_4)\rangle+\langle U(\mathbf{x}_1)U(\mathbf{x}_3)\rangle\langle U(\mathbf{x}_2)U(\mathbf{x}_4)\rangle \\ &-\langle U(\mathbf{x}_1)U(\mathbf{x}_4)\rangle\langle U(\mathbf{x}_2)U(\mathbf{x}_3)\rangle \end{aligned} \tag{4.53}$$

Since we perform integration over variables x_1, x_2, x_3, x_4 which are therefore interchangeable, four point correlator is made up of just two different contributions

$$\begin{aligned}I_1 &= \int \prod_{i=1}^{4} \mathsf{d}^2\mathbf{x}_i \langle \mathcal{V}_{\alpha_1}(\mathbf{x}_1)\mathcal{V}_{\alpha_1}(\mathbf{x}_2)\mathcal{V}_{-\alpha_1}(\mathbf{x}_3)\mathcal{V}_{-\alpha_1}(\mathbf{x}_4)\rangle \\ &\times \langle \mathcal{V}_{\alpha_2}(\mathbf{x}_1)\mathcal{V}_{\alpha_2}(\mathbf{x}_2)\mathcal{V}_{-\alpha_2}(\mathbf{x}_3)\mathcal{V}_{-\alpha_2}(\mathbf{x}_4)\rangle \\ &= \int \prod_{i=1}^{4} \frac{\mathsf{d}^2 z_i}{|z_i|^2} \frac{|z_2-z_1|^2|z_4-z_3|^2}{|z_3-z_1|^2|z_3-z_2|^2|z_4-z_1|^2|z_4-z_2|^2} \frac{|z_1|^{2p_+}|z_2|^{2p_+}}{|z_3|^{2p_+}|z_4|^{2p_+}}\end{aligned} \tag{4.54}$$

$$\begin{aligned}I_2 &= \int \prod_{i=1}^{4} \mathsf{d}^2\mathbf{x}_i \langle V_{\alpha_1}(\mathbf{x}_1)V_{\alpha_1}(\mathbf{x}_2)V_{-\alpha_1}(\mathbf{x}_3)V_{-\alpha_1}(\mathbf{x}_4)\rangle \\ &\times \langle V_{\alpha_2}(\mathbf{x}_1)V_{-\alpha_2}(\mathbf{x}_2)V_{\alpha_2}(\mathbf{x}_3)V_{-\alpha_2}(\mathbf{x}_4)\rangle \\ &= \int \prod_{i=1}^{4} \frac{\mathsf{d}^2 z_i}{|z_i|^2} \frac{|z_3-z_1|^{2\delta}|z_4-z_2|^{2\delta}}{|z_2-z_1|^{2\delta}|z_4-z_3|^{2\delta}|z_3-z_2|^2|z_4-z_1|^2} \frac{|z_1|^{2p_+}|z_2|^{2p_-}}{|z_3|^{2p_-}|z_4|^{2p_+}}\end{aligned} \tag{4.55}$$

Then contribution of (unconnected) four point correlator can be computed as

$$\int \frac{\mathsf{d}^2 z_1}{|z_1|^2}\frac{\mathsf{d}^2 z_2}{|z_2|^2}\frac{\mathsf{d}^2 z_3}{|z_3|^2}\frac{\mathsf{d}^2 z_4}{|z_4|^2} \langle U(\mathbf{x}_1(z_1))U(\mathbf{x}_2(z_2)U(\mathbf{x}_3(z_3))U(\mathbf{x}_4(z_4))\rangle = C_2^4(2I_1+4I_2) \tag{4.56}$$

While we can compute full answer, including contractions, in straightforward fashion it is more elegant to note that of among all terms contributing to second order correction only I_2 depends on δ (assuming $p_\pm$ are fixed). Then, subtracting answer for $\delta = 0$

$$\begin{aligned}e_2(\delta) &= e_2(0) + 2 \int \prod_{i=1}^{3} \frac{\mathsf{d}^2 z_i}{2\pi} \; |z_1|^{-1+2p_1+2p_2} \; |z_2|^{-1-2p_1+2p_2} \; |z_3|^{-1-2p_1-2p_2} \\ &\times \left(\left| \frac{(z_1-z_2)(1-z_3)}{(z_3-z_2)(1-z_1)} \right|^{2\delta} - 1 \right) \big| (1-z_2)(z_1-z_3) \big|^{-2} .\end{aligned} \tag{4.57}$$

This is a threefold integral, but it can be reduced to just one integration in several steps. Firstly, we express z_2 via $\zeta = \frac{(1-z_1)(z_2-z_3)}{(1-z_2)(z_1-z_3)}$, and integrate over z_1 with help of the following identity

$$\begin{aligned}&\int \frac{\mathsf{d}^2 z}{\pi} \, |z|^{-1+2p_1+2p_2} \, |1-z|^{-1+2p_1-2p_2} \, |z-w|^{-1-2p_1+2p_2} \\ &\qquad = |w|^{-1+2p_2} \; |1-w| \; \tau_{p_1 p_2}\big(\tfrac{1}{1-w}\big)\end{aligned} \tag{4.58}$$

Here function $\tau_{p_1p_2}$ is defined as

$$\begin{aligned}\tau_{p_1p_2}(\zeta) = & \frac{\Omega(-p_1,p_2)}{2p_1}\ |\zeta|^{1-2p_1}|1-\zeta|^{1+2p_2} \\ & \times \left|{}_2F_1\left(\tfrac{1}{2}-p_1+p_2,\tfrac{1}{2}-p_1+p_2,1-2p_1;\zeta\right)\right|^2 \\ & - \frac{\Omega(p_1,p_2)}{2p_1}\ |\zeta|^{1+2p_1}|1-\zeta|^{1-2p_2} \\ & \times \left|{}_2F_1\left(\tfrac{1}{2}+p_1-p_2,\tfrac{1}{2}+p_1-p_2,1+2p_1;\zeta\right)\right|^2 ,\end{aligned} \tag{4.59}$$

and $\Omega(p_1,p_2)$ reads

$$\Omega(p_1,p_2) = \frac{\Gamma(\frac{1}{2}+p_1-p_2)\Gamma(\frac{1}{2}+p_1+p_2)}{\Gamma(\frac{1}{2}-p_1-p_2)\Gamma(\frac{1}{2}-p_1+p_2)}\ \frac{\Gamma(1-2p_1)}{\Gamma(1+2p_1)} . \tag{4.60}$$

Identity (4.58) can be verified by checking that both sides satisfy same Fuchs type differential equations both in w and $\bar{w}$. Finally,we make use of nontrivial identity

$$\big(\tau_{p_1p_2}(\zeta)\big)^2 = |\zeta|^2 \int \frac{\mathsf{d}^2 z}{\pi|z|^2}\ \left|\frac{1-\zeta z}{z(z-\zeta)}\right|^{2p_1}\ \frac{\tau_{p_1p_2}\big(X(z)\big)}{|X(z)|} , \tag{4.61}$$

where $X(z) = \frac{(\zeta-z)(1-\zeta z)}{\zeta(1-z)^2}$.

$$e_2(\delta) = e_2(0) + \frac{1}{4\pi} \int \frac{\mathsf{d}^2\zeta}{|\zeta|^2|1-\zeta|^2}\ \Big(|\zeta|^{-2\delta}|1-\zeta|^{2\delta}-1\Big)\ \tau^2_{p_1p_2}(\zeta) . \tag{4.62}$$

The author does not have an independent proof of (4.61). This result was originally obtained from expansion of exact expression for scaling function obtained via ODE/IM correspondence (see chapters 6,7) and verified numerically. To compute $e_2(0)$ (or, indeed, any of $e_n(0)$) we expand the scaling function of free fermions $\mathfrak{F}_0 = \mathfrak{F}|_{\delta=0}$ which is known explicitly.

$$\mathfrak{F}_0(r,\mathbf{k}) = \mathfrak{f}(r,k_+) + \mathfrak{f}(r,k_-) \tag{4.63}$$

From results of previous chapter (see eq.(3.50)) we have

$$\mathfrak{f}\big(r,k\big) = -\frac{r}{2\pi^2} \int_{-\infty}^{\infty} \mathsf{d}\theta\, \mathsf{e}^{\theta}\ \log\Big[\Big(1+\mathsf{e}^{2\pi \mathrm{i}k}\mathsf{e}^{-r\cosh(\theta)}\Big)\Big(1+\mathsf{e}^{-2\pi \mathrm{i}k}\mathsf{e}^{-r\cosh(\theta)}\Big)\Big] . \tag{4.64}$$

Expanding it in ρ we get

$$\begin{aligned} e_n(0) \ &= \ -2\,\delta_{n,1} + \frac{(-1)^n\, n}{4^{n-1}(n!)^2}\ \Big(\psi^{(2n-2)}\big(\tfrac{1}{2}+p_1+p_2\big) + \psi^{(2n-2)}\big(\tfrac{1}{2}-p_1-p_2\big) \\ &+ \ \psi^{(2n-2)}\big(\tfrac{1}{2}+p_1-p_2\big) + \psi^{(2n-2)}\big(\tfrac{1}{2}-p_1+p_2\big)\Big) . \end{aligned} \tag{4.65}$$

Exact computation of vacuum energy via Bethe Ansatz

5.1 Coordinate Bethe Ansatz for Bukhvostov-Lipatov model

We start with *bare* Lagrangian of Bukhvostov-Lipatov model

$$\mathcal{L} = \sum_{a=\pm} \bar{\Psi}_a \big(\mathrm{i}\gamma^\mu \partial_\mu - M_0\big) \Psi_a - g_0 \left(\bar{\Psi}_+ \gamma^\mu \Psi_+\right) \left(\bar{\Psi}_- \gamma_\mu \Psi_-\right) , \tag{5.1}$$

where fermionic fields Ψ obey the following quasiperiodic boundary conditions

$$\Psi_\pm(t, x+R) = -\mathsf{e}^{2\pi \mathrm{i} p_\pm}\, \Psi_\pm(t,x) \ , \qquad \bar{\Psi}_\pm(t, x+R) = -\mathsf{e}^{-2\pi \mathrm{i} p_\pm}\, \bar{\Psi}_\pm(t,x) \ , \tag{5.2}$$

with *bare* twists p_+, p_-. Each field is a two-component Dirac spinor:

$$\Psi_a(x) = \begin{pmatrix} \psi_a^+(x) \\ \psi_a^-(x) \end{pmatrix} . \tag{5.3}$$

The model (5.1) can be solved exactly using the coordinate Bethe ansatz. In this section we present the derivation of Bethe ansatz equations originally performed by Boukhvostov and Lipatov in [48]. The only modification is the use of more general twisted boundary conditions (5.2) as opposed to antiperiodic boundary condition used in [48]. Our primary contribution, which will be discussed in subsequent sections, is the construction of suitable regularizations as well as analytical and numerical analysis of the solutions.

We start by performing second quantisation to obtain Hamiltonian:

$$\widehat{H} = \int \mathsf{d}x \,\Bigg(\sum_{a=\pm} (-\mathrm{i}\Psi_a^\dagger \sigma_3\,\partial_x \Psi_a + M_0 \Psi_a^\dagger \sigma_1 \Psi_a) \\ + g_0\,(\Psi_+^\dagger \Psi_+)(\Psi_-^\dagger \Psi_-) - g_0 (\Psi_+^\dagger \sigma_3 \Psi_+)(\Psi_-^\dagger \sigma_3 \Psi_-) \Bigg)\,. \tag{5.4}$$

Here operators $\psi_a^{s\,\dagger}, \psi_a^s$ obey standard anticommutation relations

$$\left\{\psi_{a_1}^{s_1}(x_1), \psi_{a_2}^{s_2\,\dagger}(x_2)\right\} = \delta_{a_1 a_2}\delta_{s_1 s_2}\delta(x_1 - x_2) \tag{5.5}$$

$$\left\{\psi_{a_1}^{s_1}(x_1), \psi_{a_2}^{s_2}(x_2)\right\} = \left\{\psi_{a_1}^{s_1\,\dagger}(x_1), \psi_{a_2}^{s_2\,\dagger}(x_2)\right\} = 0\,. \tag{5.6}$$

Then it is straightforward to check that operators of number of quasiparticles of given type $\mathcal{N}_+$, $\mathcal{N}_-$ and the operator of total number of quasiparticles $\mathcal{N}$

$$\widehat{\mathcal{N}}_\pm = \int \mathsf{d}x\ \Psi_\pm^\dagger \Psi_\pm\,, \qquad [\,\widehat{\mathcal{N}}_+, \widehat{\mathcal{N}}_-\,] = [\,\widehat{\mathcal{N}}_\pm, \widehat{H}\,] = 0\,, \qquad \widehat{\mathcal{N}} = \widehat{\mathcal{N}}_+ + \widehat{\mathcal{N}}_- \tag{5.7}$$

commute with Hamiltionian and with each other. Then the Fock space can be split into subspaces with fixed number of quasiparticles of each type labeled by eigenvalues of $\mathcal{N}_+$, $\mathcal{N}_-$. An arbitrary state with given total number of quasiparticles $\mathcal{N}$ can be represented as

$$|\Phi\rangle = \sum_{a_1,\ldots,a_{\mathcal{N}}} \sum_{s_1,\ldots,s_{\mathcal{N}}} \int \prod_{i=1}^{\mathcal{N}} \mathsf{d}x_i\ \chi_{a_1 a_2 \ldots a_{\mathcal{N}}}^{s_1 s_2 \ldots s_{\mathcal{N}}}(x_1, x_2, \ldots, x_{\mathcal{N}})\ \psi_{a_1}^{s_1\,\dagger}(x_1) \cdots \psi_{a_{\mathcal{N}}}^{s_{\mathcal{N}}\,\dagger}(x_{\mathcal{N}})\ |0\rangle\,. \tag{5.8}$$

Then Schrödinger equation

$$\hat{H}|\Phi\rangle = E|\Phi\rangle \tag{5.9}$$

reduces to the following PDE for "wavefunction" χ:

$$\sum_{i=1}^{\mathcal{N}} \Big(-\mathrm{i}\sigma_3^{(i)} \frac{\partial}{\partial x_i} + M_0 \sigma_1^{(i)} \Big) \boldsymbol{\chi} \\ + \frac{g_0}{2} \sum_{j=1}^{\mathcal{N}} \sum_{i<j} \delta(x_i - x_j) \big(1 - \sigma_3^{(i)}\sigma_3^{(j)}\big)\big(1 - \tau_3^{(i)}\tau_3^{(j)}\big) \boldsymbol{\chi} = E\,\boldsymbol{\chi} \tag{5.10}$$

Quasiperiodic boundary condition for wavefunction following from (5.2),(5.3) and

(5.8) are

$$\chi^{s_1\dots s_j\dots s_{\mathcal{N}}}_{a_1\dots a_j\dots a_{\mathcal{N}}}(x_1,\dots,x_j,\dots,x_{\mathcal{N}}) = -\mathrm{e}^{-2\pi \mathrm{i} p_{a_j}}\,\chi^{s_1\dots s_j\dots,s_{\mathcal{N}}}_{a_1\dots a_j\dots a_{\mathcal{N}}}(x_1,\dots,x_j+R,\dots,x_{\mathcal{N}}) \quad (\forall j\,) \tag{5.11}$$

Bukhvostov and Lipatov in [48], proposed the following ansatz for wavefunction

$$\chi^{s_1\dots s_{\mathcal{N}}}_{a_1\dots a_{\mathcal{N}}}(x_1,\dots,x_{\mathcal{N}})\Big|_{x_1<x_2<\dots<x_{\mathcal{N}}} = \sum_Q A^{q_1 q_2\dots q_{\mathcal{N}}}_{a_1 a_2\dots a_{\mathcal{N}}} \prod_{i=1}^{\mathcal{N}} u_{\theta_{q_i}}(x_i,s_i) \tag{5.12}$$

where $Q=\{q_1,q_2,...,q_{\mathcal{N}}\}$ are permutations of $\{1,2,...,\mathcal{N}\}$ and u_θ is a solution in sector $\mathcal{N}=1$

$$u_\theta(x,s) = \mathrm{i}\, s\, \mathrm{e}^{\mathrm{i}x\, M_0\, \sinh\theta - \mathrm{i}s\,\theta/2}\,, \tag{5.13}$$

Parameter θ appearing in definition (5.13) is called *rapidity* of the quasiparticle. It parametrises its energy and momentum as

$$p = M_0 \sinh\theta\,, \qquad E = -M_0\cosh\theta \tag{5.14}$$

Coefficients A^Q in (5.12) do not depend on coordinates or spins, but may depend on flavours and rapidities of quasiparticles. Flavour indices a_i can take values ± 1. Wavefunction outside the domain $0 < x_1 < x_2 < ... < x_{\mathcal{N}}$ can be computed from complete antisymmetry of fermionic wavefunction w.r.t. permutation of arguments $(x_i, s_i, a_i) \leftrightarrow (x_j, s_j, a_j)$. Then eigenstates are completely parametrised by the set of tensors A^Q. If we plug (5.12) for two-particle sector into Schrödinger equation (5.10) and take into account antisymmetry of fermionic wavefunction

$$\chi^{s_1\dots s_i s_{i+1}\dots s_{\mathcal{N}}}_{a_1\dots a_i a_{i+1}\dots a_{\mathcal{N}}}(x_1,\dots,x_i,x_{i+1},\dots,x_{\mathcal{N}}) = -\chi^{s_1\dots s_{i+1} s_i\dots,s_{\mathcal{N}}}_{a_1\dots a_{i+1} a_i\dots a_{\mathcal{N}}}(x_1,\dots,x_{i+1},x_i,\dots,x_{\mathcal{N}}) \tag{5.15}$$

and integrate in x_2 from $x_1-\varepsilon$ to $x_1+\varepsilon$ we shall get the following necessary condition for wavefunction (5.8) being an eigenfunction of Hamiltonian (5.10)

$$A^{12}_{a_1 a_2} = \sum_{a_1',a_2'} S^{a_1' a_2'}_{a_1 a_2}(\theta_1-\theta_2) A^{21}_{a_2' a_1'} \tag{5.16}$$

where matrix S is given by

$$\begin{aligned} S^{+-}_{+-}(\theta) &= S^{-+}_{-+}(\theta) = -\frac{\sinh(\theta)}{\sinh(\theta - i\pi\delta)}\,, & S^{-+}_{+-}(\theta) &= S^{+-}_{-+}(\theta) = \frac{\sinh(\mathrm{i}\pi\delta)}{\sinh(\theta-\mathrm{i}\pi\delta)}\,,\\ S^{++}_{++}(\theta) &= S^{--}_{--}(\theta) = -1\,, & \delta &= \frac{2}{\pi}\arctan(g_0)\,. \end{aligned} \tag{5.17}$$

It is familiar R-matrix of six-vertex model [50] which is known to satisfy Yang-Baxter equation

$$\sum_{a',b',c'} S^{a'b'}_{a\ b}(\theta_{12})\, S^{a''c'}_{a'\ c}(\theta_{23})\, S^{b''c''}_{b'\ c'}(\theta_{13}) = \sum_{a',b',c'} S^{b'c'}_{b\ c}(\theta_{23})\, S^{a'c''}_{a\ c'}(\theta_{13})\, S^{a''b''}_{a'\ b'}(\theta_{12}) \tag{5.18}$$

Yang-Baxter equation implies that

$$A^{q_1\,\ldots\, q_j q_{j+1}\ldots q_{\mathcal{N}}}_{a_1\ldots\, a_j a_{j+1}\ldots a_{\mathcal{N}}} = \sum_{a'_j a'_{j+1}} S^{a'_j a'_{j+1}}_{a_j a_{j+1}}(\theta_{q_j} - \theta_{q_{j+1}})\, A^{q_1\ldots\, q_{j+1} q_j \ldots q_{\mathcal{N}}}_{a_1\ldots\, a'_{j+1} a'_j \ldots a_{\mathcal{N}}}\,. \tag{5.19}$$

Relation (5.19) can be used to compute all tensors A^Q if just one of them, p.e. $A^{12\ldots\mathcal{N}}$ is known. The latter therefore fully parametrizes the states of the form (5.12). Tensors A^Q giving corresponding to such states (which are not necesserily true eigenstates) form a linear space of dimension $2^{\mathcal{N}}$, which can be represented as a tensor product $V^{\otimes\mathcal{N}}$, where V is a two-dimensional vector space. We can think of V either as a space of one-dimensional tensors, or as a fundamental representation of $\mathfrak{sl}_2$, i.e. space of states of a single spin. We introduce the following basis in the space V:

$$|\uparrow\rangle = \delta_{a,+}\,, \qquad |\downarrow\rangle = \delta_{a,-} \tag{5.20}$$

Then to each state of the Bukhvostov-Lipatov model of the form (5.12) we can associate a state of spin chain with $\mathcal{N}$ sites. Periodic boundary conditions (5.11) impose the following constraint on coefficients A:

$$A^{j\ 1\ \ldots\ \not{j}\ \ldots\ \mathcal{N}}_{a_j a_1 \ldots \not{a}_j \ldots\, a_{\mathcal{N}}} = (-)^{\mathcal{N}}\, \mathsf{e}^{-2\pi \mathrm{i} p_{a_j}}\, \mathsf{e}^{\mathrm{i} M_0 R \sinh\theta_j}\, A^{1\ \ldots\ \not{j}\ \ldots\ \mathcal{N}\ j}_{a_1 \ldots \not{a}_j \ldots\, a_{\mathcal{N}}\, a_j} \tag{5.21}$$

where notation $1 \ldots \not{j} \ldots \mathcal{N}$ stands for a sequence of integers from 1 to $\mathcal{N}$ with j omitted, i.e. $j-1$ is followed by $j+1$. Systematically applying (5.19) to the left hand side of equation (5.21) till tensors A in both sides correspond to the same permutation we can rewrite this condition as

$$\widehat{T}(\theta_j)\,|\Phi\rangle = -\mathsf{e}^{-2\pi \mathrm{i} p_{a_j}}\, \mathsf{e}^{\mathrm{i} M_0 R \sinh\theta_j}\,|\Phi\rangle \qquad (j = 1, \ldots, \mathcal{N}) \tag{5.22}$$

where operator $\widehat{T}$, known as *transfer matrix* is given by

$$\widehat{T}(\theta) = \mathrm{Tr}\widehat{\mathcal{T}}(\theta) \tag{5.23}$$

where operator $\widehat{\mathcal{T}}$, called *monodromy matrix*, is given by

$$\|\,\widehat{\mathcal{T}}^{c'}_{c}(\theta)\,\|^{a'_1 a'_2 \ldots a'_{\mathcal{N}}}_{a_1 a_2 \ldots a_{\mathcal{N}}} = (-1)^{\mathcal{N}} \sum_{b_2,\ldots,b_{\mathcal{N}}} \mathsf{e}^{2\pi \mathrm{i} p_c} S^{b_2 a'_1}_{c a_1}(\theta-\theta_1) S^{b_3 a'_2}_{b_2 a_2}(\theta-\theta_2) \cdots S^{c' a'_{\mathcal{N}}}_{b_{\mathcal{N}} a_{\mathcal{N}}}(\theta-\theta_{\mathcal{N}})\,, \tag{5.24}$$

and trace in (5.23) is computed only w.r.t. indices c, c' in (5.24), which refer to the two-dimensional *auxiliary space*. On the other hand indices a, a' are said to refer to *quantum space*, which is a space of tensors A^Q. The terminology comes from the theory of integrable spin chains and lattice models of statistical mechanics, where similar objects naturally appear. As a consequence of Yang-Baxter equation operators $\widehat{T}$ of different arguments commute with each other. The problem is now reduced to finding eigenvectors and eigenvalues of transfer-matrix (5.23). This problem is well-known in the literature on integrable systems and can be solved by means of algebraic Bethe Ansatz. We write monodromy matrix as

$$\widehat{\mathcal{T}}(\theta) = \begin{pmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{pmatrix} \tag{5.25}$$

where $A(\theta)$, $B(\theta)$, $C(\theta)$, $D(\theta)$ are linear operators acting on the space $V^{\mathcal{N}}$ which is a space of states of spin chain with $\mathcal{N}$ sites. From Yang-Baxter equation (5.18) we get the following commutation relations

$$\begin{aligned} B(u)B(v) &= B(v)B(u) \qquad (5.26) \\ A(u)B(v) &= \frac{\sinh(v-u)}{\sinh(v-u-\mathrm{i}\pi\delta)} B(v)A(u) + \frac{\sinh(\mathrm{i}\pi\delta)}{\sinh(v-u-\mathrm{i}\pi\delta)} B(u)A(v) \\ D(u)B(v) &= \frac{\sinh(v-u)}{\sinh(v-u-\mathrm{i}\pi\delta)} B(v)D(u) + \frac{\sinh(\mathrm{i}\pi\delta)}{\sinh(v-u-\mathrm{i}\pi\delta)} B(u)D(v) \end{aligned}$$

Consider *Bethe vacuum state* (not to be confused with either *bare vacuum state* $|0\rangle$ or physical ground state of the system $|\Omega\rangle$):

$$|\omega\rangle = |\uparrow\uparrow \ldots \uparrow\uparrow\rangle \tag{5.27}$$

It is easy to see that it is an eigenvector of $\widehat{T}(\theta)$:

$$\widehat{T}(\theta)|\omega\rangle = \left(\mathrm{e}^{2\pi \mathrm{i} p_+} \prod_{i=1}^{\mathcal{N}} \frac{\sinh(\theta - \theta_i)}{\sinh(\theta - \theta_i - \mathrm{i}\pi\delta)} + \mathrm{e}^{2\pi \mathrm{i} p_-} \right) |\omega\rangle \tag{5.28}$$

The eigenvectors are sought in the form

$$|\Phi\rangle = \prod_{i=1}^{\mathcal{N}_+} B(u_i)|\omega\rangle \tag{5.29}$$

Here complex numbers u_i are additional parameters which will be fixed by condition that $|\Phi\rangle$ is an actual eigenstate of the system. Each application of operator B increases $\mathcal{N}_+$ by one while preserving $\mathcal{N}$. Numbers u_i play the role of rapidities of magnons in virtual spin chain of length $\mathcal{N}$. In the Bukhvostov-Lipatov model they carry information about flavour structure of the given eigenstate of the QFT, however it is difficult to associate them to any kind of quasiparticle since they do not carry any physical momentum or energy. We shall refer to both u_k and θ_j as *Bethe roots*. If we act on vector (5.29) by operator $\widehat{T}(\theta) = A(\theta) + D(\theta)$ we get, thanks to commutation relations (5.26)

$$\widehat{T}(\theta)|\Phi\rangle = T(\theta)|\Phi\rangle + \sum_{j=1}^{\mathcal{N}_+} X_j(\theta, \{\theta_k, u_l\}) A(u_j) \prod_{i\neq j} B(u_i) B(\theta)|\omega\rangle \tag{5.30}$$

where X_j are some scalar coefficients. If vector (5.29) is an eigenvector all coefficients X_j must vanish, which leads to the following system of equations:

$$-1 \;=\; \mathsf{e}^{-4\pi \mathrm{i} p_1} \prod_{\ell'} \frac{\sinh\left(u_\ell - u_{\ell'} + \mathrm{i}\pi\delta\right)}{\sinh\left(u_\ell - u_{\ell'} - \mathrm{i}\pi\delta\right)} \prod_{\jmath} \frac{\sinh\left(u_\ell - \theta_\jmath - \frac{\mathrm{i}\pi}{2}\delta\right)}{\sinh\left(u_\ell - \theta_\jmath + \frac{\mathrm{i}\pi}{2}\delta\right)} . \tag{5.31}$$

Corresponding eigenvalue reads

$$\begin{aligned} T(\theta) \;&= \mathsf{e}^{2\pi \mathrm{i}\, p_+} \prod_{j=1}^{\mathcal{N}} \frac{\sinh(\theta - \theta_j)}{\sinh(\theta - \theta_j - \mathrm{i}\pi\delta)} \prod_{k=1}^{\mathcal{N}_+} \frac{\sinh(\theta - u_k - \frac{3\mathrm{i}\pi}{2}\delta)}{\sinh(\theta - u_k - \frac{i\pi}{2}\delta)} \\ &+ \mathsf{e}^{2\pi \mathrm{i}\, p_-} \prod_{k=1}^{\mathcal{N}_+} \frac{\sinh(\theta - u_k + \frac{\mathrm{i}\pi}{2}\delta)}{\sinh(\theta - u_k - \frac{\mathrm{i}\pi}{2}\delta)} , \end{aligned} \tag{5.32}$$

and periodicity conditions (5.21) give rise to second set of Bethe Equations

$$-1 \;=\; \mathsf{e}^{2\pi \mathrm{i}(p_1 - p_2)}\, \mathsf{e}^{\mathrm{i} M_0 R \sinh\theta_\jmath} \prod_{\ell} \frac{\sinh\left(\theta_\jmath - u_\ell - \frac{\mathrm{i}\pi}{2}\delta\right)}{\sinh\left(\theta_\jmath - u_\ell + \frac{\mathrm{i}\pi}{2}\delta\right)} . \tag{5.33}$$

The Lagrangian (5.1) is symmetric w.r.t. change of variables $\Psi_+ \leftrightarrow \Psi_-$, which means that for the ground state we must have

$$\mathcal{N}_+ - \mathcal{N}_- = 0 \tag{5.34}$$

We shall use the following notation:

$$N = \mathcal{N}_+ = \mathcal{N}_- , \qquad \mathcal{N} = 2N \tag{5.35}$$

5.1.1 Bethe Ansatz for massless case

It is not difficult to modify the solution above for the case $M_0 = 0$. Then solution in one-particle sector (5.13) gets replaced by

$$u_\theta(x,s) = \mathrm{i}\, s\, \exp\left(\pm\frac{2\pi \mathrm{i} x}{R}\mathsf{e}^\theta - \mathrm{i}s\,\theta/2\right). \tag{5.36}$$

The energy and the momentum of such quasiparticle can be expressed as

$$p_\theta = \pm\frac{2\pi\mathsf{e}^\theta}{R}, \qquad E_\theta = -\frac{2\pi\mathsf{e}^\theta}{R} \tag{5.37}$$

Sign $\pm$ distinguish right-moving and left-moving particles. Right-moving and left-moving particles do not interact since interaction term in (5.10) can be rewritten as

$$1 + \sigma_3^{(i)}\sigma_3^{(j)} = 2P_L^{(i)}P_L^{(j)} + 2P_R^{(i)}P_R^{(j)} \tag{5.38}$$

where P_R and P_L are projectors onto right-moving and left-moving states. Wavefunction (5.8) then can be decomposed as a tensor product of wavefunctions describing right and left sectors. If we make ansatz (5.12) for right-moving sector only with new one-particle solutions, we will get the same relations (5.16), (5.17). Consequently, right-moving and left-moving sectors are described by separate systems of Bethe equations. The only modification to the Bethe Ansatz equations (5.33),(5.31) is replacement

$$M_0 R \sinh\theta \mapsto 4\pi\mathsf{e}^\theta \tag{5.39}$$

5.2 Plain cutoff regularization

Consider Bethe Ansatz equations for vacuum state. The total number of quasiparticles $\mathcal{N}$ remains unspecified. But since ground state is, by definition, an eigenstate with lowest energy, and addition of new quasiparticles reduces total energy, we expect the vacuum state to contain infinite number of quasiparticles. This expectation is supported by consideration of non-interacting ($\delta = 0$) theory, where the vacuum state is a filled Dirac sea of fermions with momenta

$$Rp_n = 2\pi(n + \tfrac{1}{2} + p_\pm) \;\; p_n = M_0 \sinh\theta \tag{5.40}$$

The energy of this state

$$E = -M_0 \sum_{n=-\infty}^{\infty} \sqrt{\frac{4\pi^2(n+\frac{1}{2}+p_\pm)^2}{{M_0}^2R^2} + 1} \tag{5.41}$$

is, thus, negative and divergent. However, this energy is not a physical observable: we can always shift all the energy levels simultaneously by arbitrary constant. On the other hand, the scaling function

$$\mathfrak{F} = \frac{R}{\pi}(E - R\mathcal{E}) \tag{5.42}$$

is well defined and was computed previously(3.50). This solution is fully consistent with $\delta \to 0$ limit of Bethe ansatz equations (5.33),(5.31) which in this case become decoupled and can be solved separately.

We expect qualitatively similar structure of the ground state of coupled theory. However, it is unclear how to solve the infinite system of Bethe ansatz equations in coupled case. The way to proceed is to introduce some regularization and give arguments that relevant quantities do not depend on chosen regularization scheme.

The simplest possible way to regularize the system is to require that $|\theta_k| < \Theta$. Equivalently, we can fix the number of roots N and to seek the lowest energy configuration with this number of roots. The cutoff in this case would be the largest rapidity among the solutions. That is the original approach used by Bukhvostov and Lipatov in [48].

To facilitate the solution we write both equations in logarithmic form:

$$m_\jmath = \frac{1}{2} + p_1 - p_2 + \frac{1}{2\pi} r\cos(\pi\delta/2)\sinh\theta_\jmath + \sum_\ell \phi_{2\delta}(\theta_\jmath - u_\ell)\,, \tag{5.43a}$$

$$\overline{m}_\ell = \frac{1}{2} - 2p_1 - \sum_{\ell'} \phi_{4\delta}(u_\ell - u_{\ell'}) + \sum_\jmath \phi_{2\delta}(u_\ell - \theta_\jmath)\,. \tag{5.43b}$$

In (5.43) we use the notation $r = MR$, and the symbol ϕ_α stands for

$$\phi_\alpha(\theta) = \frac{1}{2\pi\mathrm{i}}\log\left[\frac{\sinh\left(\frac{\mathrm{i}\pi}{4}\,\alpha - \theta\right)}{\sinh\left(\frac{\mathrm{i}\pi}{4}\,\alpha + \theta\right)}\right] \tag{5.44}$$

with the phase of the logarithm chosen in such way that $\phi_\alpha(0) = 0$. The integer phases $\{m_\jmath\}$ and $\{\overline{m}_\ell\}$ play the rôle of quantum numbers, which uniquely characterise solutions of the BAE.

We used NAG routine C05RBF [1] to solve the system (5.43a-5.43b) numerically.

[1]NAG Fortran Library, Mark 25

It requires some prescription for initial approximation, and we use

$$u_k = C_{ia} \operatorname{arcsinh}\left(\frac{2\pi}{r}\left(k - \frac{1}{2} + \frac{p_1}{a_1} + \frac{p_2}{a_2}\right)\right) \tag{5.45}$$

$$\theta_k = C_{ia} \operatorname{arcsinh}\left(\frac{\pi k}{r \cos\frac{\pi\delta}{2}}\right) \tag{5.46}$$

where coefficient C_{ia} for each case was found by trial and error (i.e. various values between 0.01 and 2 were tried in loop until the routine managed to converge). However, for large values of N (for this purpose large is $N > 200$, though it may be worse depending on chosen value of coupling constant and fermionic twists) it is increasingly difficult to find good initial approximation as density of zeroes grows indefinitely near the ends of distribution. The u roots are shown to lie on the real axis. Their position is given by asymptotic formula

$$MR \sinh u_\ell = (2\ell - 1)\pi \cos\frac{\pi\delta}{2} + O(\delta) \tag{5.47}$$

which is valid for $N \gg 1$ and $\ell \ll N$. The formula (5.47) should be understood as follows: the maximal difference between actual position of the root and its asymptotic value happens near ends of distribution and grows infinitely with N, but maximal deviation among first few roots (p.e. $|\ell| < 10$) is suppressed by large N. Even though contribution of such roots to the total vacuum energy is small compared to contribution of the tails of distribution they give largest contributions to the scaling function. The θ roots are also real and form a pairs near respective u-roots:

$$\theta_{2\ell-\frac{1}{2}\pm\frac{1}{2}} = u_\ell \pm \sqrt{\frac{\pi\delta}{r\cosh u_\ell}} \tag{5.48}$$

Equation (5.43a) considered as equation for θ roots with fixed u-roots admits extra solutions. Those extra root fall into three groups: some of them are densely packed at real axis near ends of distribution, and have phases $\ell > 2N$, others have $\operatorname{Im}\theta = \pi$. Finally, there are $2N$ roots that form 2-strings near lines $\operatorname{Im}\theta = \pi \pm \frac{1}{2}\pi\delta$. The latter can be identified with roots of the system with negative coupling $-\delta$. Even though they do not solve system (5.43a,5.43b) with any phase assignment, they are still worth considering. As we shall see in the next section, u-roots of the full unregularized model are preserved by change of sign of coupling constant and it is possible to introduce regularization scheme respecting this symmetry. For this reason we refer to those 2-strings as "negative delta" roots in [65] despite the fact that in plain cutoff regularization scheme they are not roots. For small values of ℓ (and $p_1 = p_2 = 0$) formula (5.48) provides good approximation for both positive and negative δ. However, for negative δ and $\ell \gg 1$ 2-strings can exhibit more complicated structure. In particular, for small negative values of δ the separation of zeroes from asymptotic line $\mathrm{Im}\,\delta = \pi(1-\frac{1}{2}\delta)$ can have several minima. Analysis in other regularization schemes suggests that this structure is an artifact of this particular regularization. We suppose that solution of (5.43a,5.43b) with negative δ would yield close but slightly different results for both θ and u roots, but this procedure due to technical difficulties was not carried out except for small values of N. For positive δ the phase assignment reads

$$m_{\mathfrak{J}} = \mathfrak{J} \qquad \overline{m}_\ell = \ell \tag{5.49}$$

We have verified the following large-N asymptotic expansion for vacuum energy

$$\frac{ER}{\pi} = \varepsilon_\infty N^2 + \bar{\varepsilon}_\infty r^2 \log(\frac{4N}{r}) + O(1) \tag{5.50}$$

where

$$\varepsilon_\infty = -(1+\delta) \qquad \bar{\varepsilon}_\infty = -\frac{\cos^2\frac{\pi\delta}{2}}{\pi^2} \tag{5.51}$$

The scaling function computed as

$$\mathfrak{F}(r,\mathbf{k}) = \lim_{\substack{N\to\infty \\ r-\text{fixed}}} \left(\frac{ER}{\pi} - \varepsilon_\infty N^2 - \bar{\varepsilon}_\infty\, r^2 \left[\log\left(4N/r\right) + C\right] - \frac{c_{eff}}{6}\right), \tag{5.52}$$

agrees with results obtained from perturbation theory in both weak coupling and short distance limits. The constant C in the above expression is non-universal. In order to make comparisons with other methods we fix it demanding that

$$\lim_{r\to\infty} \mathfrak{F} = 0 \tag{5.53}$$

The quadratic divergence in N is naturally expected: it is a manifestation of quadratic divergence in Λ in perturbation theory. Same is true for logarithmic divergence.

5.3 Lattice type regularization

The plain cutoff regularization (5.50)-(5.52) led to some insights, but overall is not convenient for either analytical or numerical treatment due to irregular behaviour near tails. We propose the following regularization scheme (it resembles strongly one used by Saleur [49]) . As a result, routine is guaranteed to converge. Let us replace the relativistic phase term $r\cos(\pi\delta/2)\sinh(\theta)$ in (5.33) by the following lattice-type expression

$$
\begin{aligned}
r\cos(\pi\delta/2)\,\sinh(\theta) &\;\to\; N\,P(\theta,\delta)\,, && (5.54)\\
P(\theta,\delta) &\;=\; 2\pi\,\phi_{(1-\delta)}\big(\tfrac{1}{2}(\theta+\Theta)\big)+2\pi\,\phi_{(1-\delta)}\big(\tfrac{1}{2}(\theta-\Theta)\big)\,, && (5.55)
\end{aligned}
$$

where $\phi_\alpha(\theta)$ is defined in (5.44). The regularization parameter Θ effectively defines the cutoff scale. Eq.(5.33) then becomes

$$\left[\frac{\mathrm{s}\big(\theta_{\jmath}+\frac{\mathrm{i}\pi}{2}(1-\delta)\big)}{\mathrm{s}\big(\theta_{\jmath}-\frac{\mathrm{i}\pi}{2}(1-\delta)\big)}\right]^N = -\mathsf{e}^{2\pi\mathrm{i}(p_1-p_2)}\prod_{\ell}\frac{\sinh\big(\theta_{\jmath}-u_\ell-\frac{\mathrm{i}\pi}{2}\delta\big)}{\sinh\big(\theta_{\jmath}-u_\ell+\frac{\mathrm{i}\pi}{2}\delta\big)}\,, \tag{5.56}$$

where

$$\mathrm{s}(\theta)=\sinh\big(\tfrac{1}{2}(\theta+\Theta)\big)\sinh\big(\tfrac{1}{2}(\theta-\Theta)\big)\ . \tag{5.57}$$

It is algebraic in variable $x_j=\mathsf{e}^{\theta_j}$, and it is immediate to see that it has $4N$ roots.

In the logarithmic form it reads

$$m_{\jmath}=\frac{1}{2}+p_1-p_2+\frac{N\,P(\theta_{\jmath},\delta)}{2\pi}\;+\sum_{\ell}\;\phi_{2\delta}(\theta_{\jmath}-u_\ell)\,. \tag{5.58}$$

Note that r.h.s. of (5.58) as a function of $\theta_{\jmath}$ has two extra branch cuts connecting points $\theta=\pm\Theta+\mathrm{i}\pi-\frac{\mathrm{i}\pi\delta}{2}$ to points $\theta=\pm\Theta+\mathrm{i}\pi+\frac{\mathrm{i}\pi\delta}{2}$, but on the real axis it is continuous. The regularised energy is defined as

$$E_N(\Theta,\delta)=2\sum_{\jmath}\mathsf{e}(\theta_{\jmath},\delta)\,,\qquad \mathsf{e}(\theta,\delta)=\phi_{(1-\delta)}\big(\tfrac{1}{2}(\theta+\Theta)\big)-\phi_{(1-\delta)}\big(\tfrac{1}{2}(\theta-\Theta)\big)\,, \tag{5.59}$$

Expressions (5.55),(5.59) can be considered as regularizations of momentum and energy respectively since for rapidities much smaller than Θ they coincide with corresponding field theory expressions

$$P(\theta,\delta)=4\mathsf{e}^{-\Theta}\cos\frac{\pi\delta}{2}\sinh\theta+O(\mathsf{e}^{-2\Theta}) \tag{5.60}$$

$$\mathsf{e}(\theta,\delta)=(1+\delta)-4\mathsf{e}^{-\Theta}\cos\frac{\pi\delta}{2}\cosh\theta+O(\mathsf{e}^{-2\Theta}) \tag{5.61}$$

Then in the scaling limit

$$r=4N\,\mathsf{e}^{-\Theta}=\text{fixed}\,,\qquad N\to\infty,\quad \Theta\to\infty\,, \tag{5.62}$$

we recover original Bethe ansatz equations (5.33),(5.31). The energy (5.59) in the scaling limit can be formally related to the energy of the continuous model as

$$NE_N(\Theta,\delta) = \frac{RE}{\pi} + 2N^2\left(1+\delta+O(\mathsf{e}^{-2\Theta})\right), \tag{5.63}$$

where quantity E appearing in the right hand side stands for

$$E = -M\cos\frac{\pi\delta}{2}\sum\cosh\theta_{\jmath} \tag{5.64}$$

The relation (5.63) is only formal as sum (5.64) is divergent and only roots such that $\theta << \Theta$ are understood to contribute to E_N in the scaling limit. In case $\delta > 0$ root pattern for u_k, $\theta_{\jmath}$ is qualitatively similar to one observed for plain cutoff regularization. Similar arguments suggest that for vacuum state they should be real. Then from Bethe ansatz equations follows that

$$\theta_{-2N+\frac{1}{2}} < \theta_{-2N+\frac{3}{2}} < \cdots < \theta_{2N-\frac{1}{2}}\,, \qquad u_{-N+\frac{1}{2}} < u_{-N+\frac{3}{2}} < \cdots < u_{N-\frac{1}{2}}\,. \tag{5.65}$$

Indeed, the function

$$\sum\phi_{2\delta}(\theta-u_j) \tag{5.66}$$

increases monotonously and varies mostly near the points $\theta = u_j$ where it increases quickly by 1. Thus, for sufficiently small δ, there are two θ-roots around each u-root. If we fix Θ and take sufficiently large N, we can approximate the "discrete densities" of roots

$$\rho_u^{(N)}\left(u_{n+\frac{1}{2}}\right) = \frac{1}{N(u_{n+1}-u_n)} \qquad \left(u_{n+1/2}\equiv\tfrac{1}{2}(u_{n+1}+u_n)\right), \tag{5.67}$$

by continuous functions

$$\rho_u(u) = \frac{1}{2\pi}\left(\frac{1}{\cosh(u+\Theta)}+\frac{1}{\cosh(u-\Theta)}\right). \tag{5.68}$$

$$\rho_\theta(\theta) = 2\,\mathrm{Re}\left(\rho_u\left(\theta+\tfrac{1}{2}\mathrm{i}\pi\delta\right)\right). \tag{5.69}$$

Then positions of roots are well approximated by

$$\begin{aligned} N\,\mathsf{z}_\theta(\theta_{\jmath}) &\approx \jmath-\frac{1}{2}+\frac{p_1}{a_1}+\frac{p_2}{a_2} && (1\ll|\jmath|\ll N) \\ N\,\mathsf{z}_u(u_\ell) &\approx \ell-\frac{1}{2}+\frac{2p_2}{a_2} && (1\ll 2|\ell|\ll N)\,, \end{aligned} \tag{5.70}$$

where we've introduced lattice zero-counting functions z_u, z_u

$$\mathsf{z}_\theta(\theta) \;=\; \int_0^\theta \rho_\theta(\theta')\,\mathsf{d}\theta' = \frac{2}{\pi}\int_0^\infty \frac{\mathsf{d}\nu}{\nu}\;\sin(\nu\theta)\;\frac{\cos(\nu\Theta)\cosh\left(\frac{\pi\nu\delta}{2}\right)}{\cosh\left(\frac{\pi\nu}{2}\right)} \tag{5.71a}$$

$$\mathsf{z}_u(u) \;=\; \int_0^u \rho_u(u')\,\mathsf{d}u' = \frac{1}{\pi}\int_0^\infty \frac{\mathsf{d}\nu}{\nu}\;\sin(\nu u)\;\frac{\cos(\nu\Theta)}{\cosh\left(\frac{\pi\nu}{2}\right)}\,. \tag{5.71b}$$

This expressions allow to establish large N behaviour of lattice energy *for fixed* Θ.

$$E_N^{(1)}(\Theta,\delta) = 2\sum_{\mathcal{J}} \mathsf{e}(\theta_{\mathcal{J}},\delta) = N\varepsilon_\infty^{(1)} + O(N^{-1}) \qquad (\delta > 0)\,, \tag{5.72}$$

The leading term can be computed in straightforward fashion

$$\varepsilon_\infty^{(1)} = \int_{-\infty}^{\infty} \mathsf{e}(\theta,\delta)\rho_\theta(\theta)d\theta \tag{5.73}$$

The answer is conveniently represented as Fourier integral

$$\varepsilon_\infty^{(1)} = \varepsilon_\infty(\Theta,\delta) = \frac{2}{\pi}\int_{-\infty}^\infty \frac{\mathsf{d}\nu}{\nu}\;\sin(2\nu\Theta)\;\frac{\sinh\left(\frac{\pi\nu(1+\delta)}{2}\right)\cosh\left(\frac{\pi\nu\delta}{2}\right)}{\sinh\left(\pi\nu\right)\cosh\left(\frac{\pi\nu}{2}\right)}\,. \tag{5.74}$$

This representation allows to compute its large Θ expansion

$$\varepsilon_\infty(\Theta,\delta) = (1+\delta) - \left(\tfrac{4}{\pi}\,\mathsf{e}^{-\Theta}\cos(\tfrac{1}{2}\pi\delta)\right)^2\,\Big(\Theta + \tfrac{1}{2} + \tfrac{1}{4}\pi\,(1+2\delta)\tan(\tfrac{1}{2}\pi\delta)\Big) + O(\mathsf{e}^{-3\Theta})\,. \tag{5.75}$$

Now let us consider case $\delta < 0$. It is convenient to change notations slightly so that δ is still positive, but in Bethe ansatz equations we make the substitution $\delta \mapsto -\delta$ and exchange p_1 with p_2. θ-roots are now complex and form 2-strings. For sufficiently large N their approximate positions are given by

$$\theta_{2\ell-\frac{1}{2}\pm\frac{1}{2}} \sim u_\ell \pm \tfrac{1}{2}\mathrm{i}\pi\delta\,, \tag{5.76}$$

Unlike 1-string, roots are not actually arranged in straight lines: they approach it closely near $\theta = \pm\Theta$ where the majority of roots are clustered, and deviate more from lines $\operatorname{Im}\theta = \pm\frac{\pi\delta}{2}$ near zero and the ends of distribution. Regularization prescription has to be modified in this case: instead of ϕ_α we shall use $\tilde{\phi}_\alpha$ such that

$$\tilde{\phi}_\alpha(\theta) = \frac{1}{2\pi\mathrm{i}}\log\frac{\sinh(\theta - \frac{\mathrm{i}\pi\alpha}{4})}{\sinh(\theta + \frac{\mathrm{i}\pi\alpha}{4})} \tag{5.77}$$

This way we ensure that 2-strings do not cross branch cuts (which are now located at) and $\tilde{\phi}_{1-\delta}$ varies continuously along the lines $\operatorname{Im}\theta = \pm\frac{\pi\delta}{2}$ Similarly to 1-strings we can compute the energy of vacuum filled with 2-strings:

$$E_N^{(2)}(\Theta,\delta) = 2\sum_{\jmath} \mathsf{e}(\theta_\jmath, -\delta)\,, \tag{5.78}$$

$$E_N^{(2)}(\Theta,\delta) = N\,\epsilon_\infty^{(2)} + O(N^{-1})\,, \qquad \varepsilon_\infty^{(2)} = \varepsilon_\infty(\Theta,-\delta)\,, \tag{5.79}$$

Remarkably, the answer for non-divergent part of the energy remains the same:

$$E_N^{(1)}(\Theta,\delta) - N\,\varepsilon_\infty^{(1)} = E_N^{(2)}(\Theta,\delta) - N\,\varepsilon_\infty^{(2)} \tag{5.80}$$

Identity (5.80) is exact and can be proven with help of integral representations for energy which will be derived in chapter 7. Let us introduce the following notations

$$\mathsf{Q}_1(\mathsf{e}^\theta) = \prod_{\jmath} \sinh\left(\tfrac{1}{2}(\theta - \theta_\jmath^{(1)})\right)\,, \qquad \mathsf{Q}_3(\mathsf{e}^\theta) = \prod_{\ell} \sinh\left(\theta - \theta_\ell^{(3)}\right)\,, \tag{5.81}$$

where $\theta^{(1)}$ are related to roots composing 1-string, and $\theta^{(3)}$ are u-roots

$$\theta_\jmath^{(1)} = \theta_\jmath + \mathrm{i}\pi, \qquad \theta_\ell^{(3)} = u_\ell\,, \tag{5.82}$$

Then we can rewrite Bethe ansatz equations as

$$-1 = \mathsf{e}^{2\pi\mathrm{i}(p_2-p_1)}\,\frac{\mathsf{f}(-\mathrm{i}\lambda_\jmath^{(1)}q^{-1})}{\mathsf{f}(\mathrm{i}\lambda_\jmath^{(1)}q)}\,\frac{\mathsf{Q}_3(\lambda_\jmath^{(1)}q)}{\mathsf{Q}_3(\lambda_\jmath^{(1)}\,q^{-1})}\,, \qquad \lambda_\jmath^{(1)} \equiv \mathsf{e}^{\theta_\jmath^{(1)}} \tag{5.83a}$$

$$-1 = \mathsf{e}^{-4\pi\mathrm{i}p_1}\,\frac{\mathsf{Q}_3(\lambda_\ell^{(3)}\,q^2)}{\mathsf{Q}_3(\lambda_\ell^{(3)}q^{-2})}\,\frac{\mathsf{Q}_1(\lambda_\ell^{(3)}q^{-1})}{\mathsf{Q}_1(\lambda_\ell^{(3)}\,q)}\,\frac{\mathsf{Q}_1(-\lambda_\ell^{(3)}q^{-1})}{\mathsf{Q}_1(-\lambda_\ell^{(3)}\,q)}\,, \qquad \lambda_\ell^{(3)} \equiv \mathsf{e}^{\theta_\ell^{(3)}}\,, \tag{5.83b}$$

where f stands for

$$\mathsf{f}(\mathsf{e}^\theta) = \left(\mathrm{s}(\theta)\right)^N\,, \qquad q = \mathsf{e}^{\frac{1}{2}\mathrm{i}\pi\delta} \tag{5.84}$$

Recall that equation (5.56) considered as equation on $\theta^{(1)}$ has $4N$ roots: we can rewrite it as equation for roots of the polynomial

$$\mathsf{P}(\lambda) = \mathsf{e}^{\mathrm{i}\pi(p_1-p_2)}\,\mathsf{f}(\mathrm{i}\lambda q)\;\mathsf{Q}_3(\lambda q^{-1}) + \mathsf{e}^{-\mathrm{i}\pi(p_1-p_2)}\,\mathsf{f}(-\mathrm{i}\lambda q^{-1})\;\mathsf{Q}_3(\lambda q) \tag{5.85}$$

Since there are $2N$ roots $\theta^{(1)}$, polynomial P must factorise as

$$\mathsf{P}(\lambda) = \mathrm{const}\;\mathsf{Q}_1(\lambda)\,\mathsf{Q}_2(-\lambda)\,, \tag{5.86}$$

where Q_2 is a polynomial of degree $2N$, and we have denoted its roots as $\theta^{(2)}$

$$\mathsf{Q}_2(\mathsf{e}^{\theta}) = \prod_{\jmath} \sinh\left(\tfrac{1}{2}(\theta - \theta^{(2)}_{\jmath})\right), \qquad \lambda^{(2)}_{\jmath} = \mathsf{e}^{\theta^{(2)}_{\jmath}}. \tag{5.87}$$

Then combining (5.85) and (5.87) we get

$$\mathsf{Q}_1(\lambda^{(3)}_{\ell} q^{\pm 1}) = \mathsf{e}^{\mp \mathrm{i}\pi(p_1 - p_2)} \mathsf{f}(\mp i \lambda^{(3)}_{\ell}) \frac{\mathsf{Q}_3(\lambda^{(3)}_{\ell} q^{\pm 2})}{\mathsf{Q}_2(-\lambda^{(3)}_{\ell} q^{\pm 1})} \tag{5.88}$$

Plugging (5.88) into equation (5.83b) we obtain

$$-1 = \mathsf{e}^{-4\pi \mathrm{i} p_2} \ \frac{\mathsf{Q}_3(\lambda^{(3)}_{\ell}\, q^{-2})}{\mathsf{Q}_3(\lambda^{(3)}_{\ell}\, q^{2})} \frac{\mathsf{Q}_2(\lambda^{(3)}_{\ell}\, q)}{\mathsf{Q}_2(\lambda^{(3)}_{\ell} q^{-1})} \frac{\mathsf{Q}_2(-\lambda^{(3)}_{\ell}\, q)}{\mathsf{Q}_2(-\lambda^{(3)}_{\ell} q^{-1})} \tag{5.89}$$

Substituting $\lambda = -\lambda^{(2)}_{\jmath}$ into (5.86),(5.85) we derive equation similar to (5.83a)

$$-1 = \mathsf{e}^{2\pi \mathrm{i}(p_1 - p_2)} \ \frac{\mathsf{f}(-\mathrm{i}\lambda^{(2)}_{\jmath} q)}{\mathsf{f}(\mathrm{i}\lambda^{(2)}_{\jmath}\, q^{-1})} \ \frac{\mathsf{Q}_3(\lambda^{(2)}_{\jmath}\, q^{-1})}{\mathsf{Q}_3(\lambda^{(2)}_{\jmath} q)} \tag{5.90}$$

Finally, expressing only $\mathsf{Q}_1(-\lambda^{(3)}_{\ell} q^{\pm 1})$ in (5.83b) via (5.88) we obtain

$$-1 = \mathsf{e}^{-2\pi \mathrm{i}(p_1 + p_2)} \ \frac{\mathsf{f}(-\mathrm{i}\lambda^{(3)}_{\ell})}{\mathsf{f}(\mathrm{i}\lambda^{(3)}_{\ell})} \ \frac{\mathsf{Q}_1(\lambda^{(3)}_{\ell} q^{-1})}{\mathsf{Q}_1(\lambda^{(3)}_{\ell}\, q)} \ \frac{\mathsf{Q}_2(\lambda^{(3)}_{\ell}\, q)}{\mathsf{Q}_2(\lambda^{(3)}_{\ell} q^{-1})} \tag{5.91}$$

Obtained system of four equations (5.90), (5.83a), (5.83b), (5.91) is excessive: indeed it has 3 closed subsystems

- I (5.83a) and (5.83b)
- II (5.90) and (5.89)
- III (5.83a), (5.90) and (5.91)

solutions of each one contains full data describing the vacuum state. Indeed, under transformation

$$\mathsf{Q}_1(\lambda) \leftrightarrow \mathsf{Q}_2(\lambda), \qquad \mathsf{Q}_3(\lambda) \leftrightarrow \mathsf{Q}_3(\lambda), \qquad p_1 \leftrightarrow p_2, \qquad q \mapsto q^{-1}, \qquad \delta \mapsto -\delta \tag{5.92}$$

system I transforms into II and vice versa, while III remains unaffected. Systems I and II describe filling of vacuum state at $\delta > 0$ and $\delta < 0$. This symmetry implies that this transformation preserves the roots of Q_3.

Exact solution via ODE/IQFT correspondence

The idea of ODE/IM correspondence is based on the fact that integrable lattice models and integrable quantum field theories are typically governed by a set of functional equations, such as Baxter's T-Q equation. It turns out that for some integrable models it is possible to implement those functional relations in a simplier system, like an ODE or PDE. The first contributions were made by Voros [86], followed by Dorey and Tateo [51], who related the T-Q equation arising in conformal field theory [87] to the relations between monodromy data of a second order ODE describing anharmonic oscillator. Soon after these results were extended and proved [52]. Further details can be found in review [58] and references therein. The appearance of such relations is a consequence of a special symmetry of the ODE. The group of transformations which preserves the ODE (but not its solutions) can be thought of as a generalisation of the concept of monodromy group for potentials that have branch cuts.

In this chapter we review the ODE/IM construction for Bukhvostov-Lipatov model, loosely following discussion in [66, 77]

6.1 Auxiliary linear problem for classical modified Sinh-Gordon equation

Consider the following PDE, which we shall refer to as *modified Sinh-Gordon equation*:

$$\partial_z \partial_{\bar{z}} \eta - \mathsf{e}^{2\eta} + \rho^4 \left|\mathcal{P}(z)\right|^2 \mathsf{e}^{-2\eta} = 0 \tag{6.1}$$

with function $\mathcal{P}(z)$ given by

$$\mathcal{P}(z) = \frac{(z_3 - z_2)^{a_1}(z_3 - z_1)^{a_2}(z_2 - z_1)^{a_3}}{(z - z_1)^{2-a_1}(z - z_2)^{2-a_2}(z - z_3)^{2-a_3}}\,, \tag{6.2}$$

where

$$a_1 = 1-\delta, \qquad a_2 = 1+\delta, \qquad a_3 = 0\,. \tag{6.3}$$

Parameter ρ is assumed to be real and non-negative. Equation (6.1) is an integrable PDE which can be transformed into Sinh-Gordon equation by change of variables

$$\eta(z,\bar z) \to \hat\eta(w,\bar w) = \eta(z,\bar z) - \tfrac{1}{2}\log\left|\rho^2\,\mathcal{P}(z)\right|, \qquad \mathsf{d}w = \rho\sqrt{\mathcal{P}(z)}\;\mathsf{d}z \tag{6.4}$$

It can be interpreted as a flatness condition for the following connection:

$$\begin{aligned} \boldsymbol{A}_z &= -\tfrac{1}{2}\,\partial_z\eta\,\sigma_3 + \left(\mathsf{e}^{\eta}\,\sigma_+ + \rho^2\,\mathsf{e}^{2\theta}\,\mathcal{P}(z)\,\mathsf{e}^{-\eta}\,\sigma_-\right) \\ \boldsymbol{A}_{\bar z} &= \tfrac{1}{2}\,\partial_{\bar z}\eta\,\sigma_3 + \left(\mathsf{e}^{\eta}\,\sigma_- + \rho^2\,\mathsf{e}^{-2\theta}\,\bar{\mathcal{P}}(\bar z)\,\mathsf{e}^{-\eta}\,\sigma_+\right), \end{aligned} \tag{6.5}$$

Parameter θ above is an additive spectral parameter, which is not defined by equation itself but can assume arbitrary complex value. Thus with any given solution we can associate an *auxiliary linear problem*

$$(\partial_z - \boldsymbol{A}_z)\,\boldsymbol{\Psi} = 0\,, \qquad (\partial_{\bar z} - \boldsymbol{A}_{\bar z})\,\boldsymbol{\Psi} = 0 \tag{6.6}$$

which is an important ingredient of solution of (6.1) by classical inverse scattering method. The auxiliary linear problem is in fact a 1-parametric family of systems of linear differential equations parametrized by value of spectral parameter. As we shall see this family as a whole carries information about all integrals of motion of modified sinh-Gordon equation computed on given solution η. However, we shall typically refer to the solution of system of differential equations for some given value of spectral parameter as to solution of auxiliary linear problem. If $(\Psi_1,\Psi_2)^T$ is a solution (of either mShG or Liouville auxiliary linear problems) such that $\Psi_1 \not\equiv 0$ then it can be parametrised as

$$\boldsymbol{\Psi} = \begin{pmatrix} \mathsf{e}^{\frac{\eta(z,\bar z)}{2}}\,\psi(z,\bar z) \\ \mathsf{e}^{-\frac{\eta(z,\bar z)}{2}}\left(\partial_z\psi(z,\bar z) + \partial_z\eta(z,\bar z)\psi(z,\bar z)\right) \end{pmatrix} \tag{6.7}$$

where function $\psi(z,\bar z)$ satisfies second order ODE

$$-\partial_z^2\psi(z,\bar z) + \left(\partial_z^2\eta(z,\bar z) - (\partial_z\eta(z,\bar z))^2\right)\psi(z,\bar z) + \rho^2\mathsf{e}^{2\theta}\mathcal{P}(z)\psi(z,\bar z) = 0 \tag{6.8}$$

If $\Psi_2 \not\equiv 0$ then there is alternative parametrization

$$\boldsymbol{\Psi} = \begin{pmatrix} \mathsf{e}^{-\frac{\eta(z,\bar z)}{2}}\left(\partial_{\bar z}\bar\psi(z,\bar z) + \partial_{\bar z}\eta(z,\bar z)\bar\psi(z,\bar z)\right) \\ \mathsf{e}^{\frac{\eta(z,\bar z)}{2}}\,\bar\psi(z,\bar z) \end{pmatrix} \tag{6.9}$$

where function $\bar{\psi}(z,\bar{z})$ satisfies second order ODE

$$-\partial_{\bar{z}}^2\bar{\psi}(z,\bar{z}) + \left(\partial_{\bar{z}}^2\eta(z,\bar{z}) - (\partial_{\bar{z}}\eta(z,\bar{z}))^2\right)\bar{\psi}(z,\bar{z}) + \rho^2\mathsf{e}^{-2\theta}\bar{\mathcal{P}}(\bar{z})\bar{\psi}(z,\bar{z}) = 0 \tag{6.10}$$

Note that $\bar{\psi}(z,\bar{z})$ in (6.9), (6.10) is **not** a complex conjugate of $\psi(z,\bar{z})$ from (6.7), (6.8). It is straightforward to show that if $\mathbf{\Psi}_+$ and $\mathbf{\Psi}_-$ are solutions admitting representation (6.7) with functions ψ_+,ψ_- respectively, then

$$\det(\mathbf{\Psi}_+,\mathbf{\Psi}_-) = \mathrm{Wr}\left[\psi_+,\psi_-\right] = \overline{\mathrm{Wr}}\left[\bar{\psi}_+,\bar{\psi}_-\right] \tag{6.11}$$

where Wronskians are defined as follows

$$\mathrm{Wr}\left[\psi_1,\psi_2\right] = \psi_2\partial_z\psi_1 - \psi_1\partial_z\psi_2\,, \qquad \overline{\mathrm{Wr}}\left[\bar{\psi}_1,\bar{\psi}_2\right] = \bar{\psi}_2\partial_{\bar{z}}\bar{\psi}_1 - \bar{\psi}_1\partial_{\bar{z}}\bar{\psi}_2 \tag{6.12}$$

In the limit $\rho \to 0$ equation (6.1) reduces to Liouville equation:

$$\partial_z\partial_{\bar{z}}\eta - \mathsf{e}^{2\eta} = 0 \tag{6.13}$$

The latter is preserved by full group of conformal transformations in two dimensions, hence we shall refer to the limit $\rho \to 0$ as *conformal limit* of the problem. As we shall see later parameter ρ can be related to the volume of Bukhvostov-Lipatov model:

$$\rho = \frac{r}{4\pi}\cos(\tfrac{1}{2}\pi\delta) \tag{6.14}$$

6.2 Conformal limit of auxiliary linear problem

Liouville equation implies that if $\eta(z,\bar{z})$ is a solution to (6.13) then

$$u_z(z) = \partial_z^2\eta(z,\bar{z}) - (\partial_z\eta(z,\bar{z}))^2 \tag{6.15}$$

does not depend on $\bar{z}$. Hence general solution of (6.8) has a form

$$\psi(z,\bar{z}) = \bar{\varphi}(\bar{z})\left(C_1\varphi_1(z) + C_2\varphi_2(z)\right) \tag{6.16}$$

where C_1, C_2 are arbitrary constants, $\bar{\varphi}(\bar{z})$ is an arbitrary antiholomorphic function and φ_1, φ_2 are two linearly independent holomorphic solutions of (6.8).

General solution to Liouville equation can be written as

$$\mathsf{e}^{-\eta} = \varphi_1(z)\bar{\varphi}_1(\bar{z}) + \varphi_2(z)\bar{\varphi}_2(\bar{z}) \tag{6.17}$$

where φ_1,φ_2 are holomorphic solutions of (6.8) at $\rho = 0$, and $\bar{\varphi}_1$,$\bar{\varphi}_2$ solutions of

(6.10) at $\rho = 0$ normalized so that

$$\mathrm{Wr}\,[\varphi_1, \varphi_2] = 1\,, \qquad \overline{\mathrm{Wr}}\,[\bar{\varphi}_1, \bar{\varphi}_2] = 1 \tag{6.18}$$

There are, however, many more ways to write down the auxillary linear problem for Liouville equation. For our purposes it is convenient to write it as

$$\begin{aligned} \boldsymbol{A}_z &= -\tfrac{1}{2}\,\partial_z \eta\,\sigma_3 + \left(\mathsf{e}^{\eta}\,\sigma_+ \,+\, \lambda^2\,\mathcal{P}(z)\,\mathsf{e}^{-\eta}\,\sigma_-\right) \\ \boldsymbol{A}_{\bar z} &= \quad \tfrac{1}{2}\,\partial_{\bar z}\eta\,\sigma_3 + \left(\mathsf{e}^{\eta}\,\sigma_-\right)\,, \end{aligned} \tag{6.19}$$

This expression can be obtained from (6.5) in the limit

$$\rho \to 0 \quad \theta \to \infty \quad \rho\,\mathsf{e}^{\theta} = \lambda = const \tag{6.20}$$

Linear problem (6.19) is not unique: indeed, we could replace $\mathcal{P}(z)$ by an arbitrary function of z and still get the same Liouville equation as consistency condition. However, different choice would lead to different symmetry properties of the auxiliary linear problem breaking the ODE/IM construction.

Now let η be a solution to (6.13) on a sphere with three punctures z_1, z_2, z_3 with the following asymptotic behaviour near punctures:

$$\mathsf{e}^{-\eta(z)} \sim |z|^2 \qquad \text{as} \qquad |z| \to \infty\,. \tag{6.21}$$

$$\eta(z) = -(1 - a_i|k_i|)\log|z - z_i| + \eta^{(i)}_{reg} + o(1) \quad \text{as} \quad |z - z_i| \to 0 \quad (i = 1, 2) \tag{6.22}$$

$$\eta(z) = -\log|z - z_3| + O(1) \qquad \text{as} \qquad |z - z_3| \to 0\,, \tag{6.23}$$

In conformal case considered here this solution was found explicitly: it can be expressed via function (4.59) that had previously appeared in context of conformal perturbation theory (4.62):

$$\eta(z) = -\log \tau_{p_1,p_2}(z) \tag{6.24}$$

The correspondence requires that parameters k_1, k_2 are same as in chapter 4, i.e. they are related to fermionic twists. Instead of parameters k_i we shall use p_i which appear naturally in CPT series. They read

$$k_\pm = k_1 \pm k_2\,, \qquad p_i = a_i|k_i|/2 \qquad (i = 1, 2)\,. \tag{6.25}$$

We do not need to specify parameters $\eta^{(i)}_{reg}$ at this point since solution is fully defined by leading terms of asymptotical expressions (6.22), (6.23). Instead, we use the occasion to introduce notation for this coefficients.

Consider equation (6.8), which we shall rewrite as

$$-\partial_z^2\varphi(z) + u_z(z)\varphi(z) + \lambda^2\mathcal{P}(z)\varphi(z) = 0 \tag{6.26}$$

where $\eta(z,\bar{z})$ is a solution of (6.13) satisfying asymptotic conditions(6.21)-(6.23) and potential u_z was defined in (6.15). From explicit expression (4.59) for solution η we can find explicit expression for potential (shortcut notation $z_{ij} = z_i - z_j$ is used):

$$u_z(z) = -\frac{z_{12}\,z_{23}\,z_{31}}{(z-z_1)(z-z_2)(z-z_3)}\left(\frac{p_1^2-\frac{1}{4}}{(z-z_1)\,z_{23}} + \frac{p_2^2-\frac{1}{4}}{(z-z_2)\,z_{31}} - \frac{1}{4(z-z_3)\,z_{12}}\right) \tag{6.27}$$

We note that it is invariant under transformations $\widehat{\Omega}_i$, which are defined as follows:

- First, we continue it analytically in z along a closed counterclockwise-oriented contour γ_i such that z_i is inside it and two other punctures are outside (to be more precise, for each z_0 we compute new potential by picking a closed contour $\gamma_i(z_0)$ passing through this point and satisfying conditions above and continuing the potential analytically along $\gamma_i(z_0)$ once).

- Second, we shift spectral parameter as $\lambda \mapsto \lambda q_i^{-1}$ with numbers q_i given by

$$q_1 = \mathsf{e}^{-\mathrm{i}\pi\delta}\,, \qquad q_2 = \mathsf{e}^{+\mathrm{i}\pi\delta}\,, \qquad q_3 = 1, \qquad q_1\,q_2\,q_3 = 1\,. \tag{6.28}$$

We can summarize the above description as

$$\widehat{\Omega}_i: \qquad z \mapsto \gamma_i \circ z\,, \qquad \bar{z} \mapsto \bar{\gamma}_i \circ \bar{z}\,, \qquad \lambda \mapsto \lambda\, q_i^{-1} \qquad (i=1,2,3) \tag{6.29}$$

A second order ODE has two linearly independent solutions. Transformations $\widehat{\Omega}_i$ map solution to solution and hence are linear transformations on the space of solutions of ODE. We introduce three bases of solutions $\varphi^{(i)} = (\varphi_-^{(i)}(z), \varphi_+^{(i)}(z))$ as eigenfunctions of transformations $\widehat{\Omega}_i$:

$$\widehat{\Omega}_i\,\varphi_\sigma^{(i)} = -\mathsf{e}^{2\pi\mathrm{i}p_i\sigma}\varphi_\sigma^{(i)} \quad i=1,2 \tag{6.30}$$

$$\widehat{\Omega}_3\,\varphi_\sigma^{(3)} = -\mathsf{e}^{2\pi\mathrm{i}\lambda\sigma}\varphi_\sigma^{(3)} \tag{6.31}$$

We normalise them so that

$$[\varphi_-^{(i)}, \varphi_+^{(i)}] = 1 \qquad i=1,2,3 \tag{6.32}$$

It is not difficult to check that such solutions exist: in the vicinity of punctures z_1, z_2 the equation reads

$$-\partial_z^2\varphi + \frac{p_i^2 - \frac{1}{4}}{(z-z_i)^2}\varphi + O(z-z_i) = 0 \tag{6.33}$$

Consequently, there are two solutions which behave near the singularity z_i as

$$\psi_\sigma^{(i)} \sim \frac{1}{\sqrt{2p_i}}(z-z_i)^{\frac{1}{2}\pm p_i}(1+O(z-z_i)) \tag{6.34}$$

Obviously, they transform under action of $\widehat{\Omega}_i$ in accordance with (6.30). Equation (6.26) looks slightly different in vicinity of z_3

$$-\partial_z^2\varphi + \frac{\lambda^2 - \frac{1}{4}}{(z-z_3)^2}\varphi + O(z-z_3) = 0 \tag{6.35}$$

It has solutions with the following asymptotic behaviour near $z = z_3$

$$\varphi_\sigma^{(3)} \sim \frac{1}{\sqrt{2\lambda}}(z-z_3)^{\frac{1}{2}\pm\lambda}(1+O(z-z_3)) \tag{6.36}$$

in agreement with (6.31).

To relate different bases we introduce connection matrices $\boldsymbol{S}$ defined by relation

$$\varphi^{(i)}\boldsymbol{S}^{(i,j)} = \varphi^{(j)} \tag{6.37}$$

Connection matrices can be expressed in terms of Wronskians of solutions

$$\mathrm{Wr}\,[\varphi_\sigma^{(i)}, \varphi_{\sigma'}^{(j)}] = \boldsymbol{S}^{(j,i)}_{-\sigma',\sigma} \tag{6.38}$$

and therefore do not depend on z. They are functions of spectral parameter λ and also depend on δ, p_1, p_2 and positions of singularities z_1, z_2, z_3. Now, using (6.37), (6.30) and (6.31) we can compute the action of transformation $\widehat{\Omega}_j$ on solutions $\varphi^{(i)}$ $(i \neq j)$

$$\widehat{\Omega}_j\,\varphi_\sigma^{(i)} \;=\; \widehat{\Omega}_j\,\varphi_{\sigma'}^{(j)}\,\boldsymbol{S}^{(j,i)}_{\sigma'\sigma}(\lambda) \;=\; -\varphi_{\sigma''}^{(i)}\,\boldsymbol{S}^{(i,j)}_{\sigma''\sigma'}(\lambda)\,\mathsf{e}^{-2\pi\mathrm{i}\sigma' p_j(\lambda)}\boldsymbol{S}^{(j,i)}_{\sigma',\sigma}(\lambda q_j^{-1}) \tag{6.39}$$

where summation over indices σ', σ'' is assumed. The shift of argument of rightmost connection matrix in (6.39) is a result of action of transformation (6.29). Connection matrices satisfy the following properties:

$$\det\big(\boldsymbol{S}^{(j,i)}(\lambda)\big) = 1\ , \quad \boldsymbol{S}^{(i,j)}(\lambda)\,\boldsymbol{S}^{(j,i)}(\lambda) = \boldsymbol{I}\ , \tag{6.40}$$

$$\boldsymbol{S}^{(i,k)}(\lambda)\ \boldsymbol{S}^{(k,j)}(\lambda)\ \boldsymbol{S}^{(j,i)}(\lambda) = \boldsymbol{I} \tag{6.41}$$

The first one follows from normalisation (6.32). Second and third properties can be verified by direct application of definition (6.37). There is also a non-trivial relation involving connection matrices at different values of spectral parameter. Since factors q_i obey

$$q_1\, q_2\, q_3 = 1 \tag{6.42}$$

composition of all three symmetries is an identity transformation:

$$\widehat{\Omega}_3 \circ \widehat{\Omega}_2 \circ \widehat{\Omega}_1 = \mathrm{Id} \tag{6.43}$$

Indeed, the contour $\gamma_3 \circ \gamma_2 \circ \gamma_1$ is contractible. Then applying both sides to the row $(\varphi_-^{(i)}, \varphi_+^{(i)})$ and using properties (6.39) and (6.41) we get

$$\boldsymbol{S}^{(i,k)}(\lambda)\ \mathsf{e}^{-2\pi \mathrm{i} p_k(\lambda)\sigma_3}\ \boldsymbol{S}^{(k,j)}(\lambda\, q_k^{-1})\ \mathsf{e}^{-2\pi \mathrm{i} p_j(\lambda\, q_k^{-1})\sigma_3}\ \boldsymbol{S}^{(j,i)}(\lambda\, q_i)\ \mathsf{e}^{-2\pi \mathrm{i} p_i(\lambda\, q_i)\sigma_3} = -\boldsymbol{I}\ . \tag{6.44}$$

Note that (6.43) is only true for that particular order of transformations and its cyclic permutation. Reversing this order would lead to non-contractible contour $\gamma_1 \circ \gamma_2 \circ \gamma_3$. Since connection matrices depend on z_1, z_2, z_3 which are not parameters of Bukhvostov-Lipatov Lagrangian definition (6.37) is ambiguous. To fix the ambiguity we consider three ODEs that are related to (6.8) by simple conformal transformations which absorb the dependence on points. Matrix elements of connection matrices for general z_1, z_2, z_3 can then be written as Wronskians of solutions of these equations multiplied by simple prefactor arising from conformal transformation. Consider three changes of variables, which map points z_1, z_2 and z_3 to $+\infty$, $\mathrm{i}\pi$ and $-\infty$ in different order (there is an essential singularity at infinity, so $-\infty$ and $+\infty$ are effectively distinct points):

$$\mathsf{e}^{x} = \frac{(z-z_3)(z_2-z_1)}{(z-z_1)(z_3-z_2)}\,, \qquad \Upsilon(x) = \varphi(z)\,\left(\frac{\mathsf{d}z}{\mathsf{d}x}\right)^{-\frac{1}{2}} \tag{6.45}$$

$$\mathsf{e}^{y} = -1-\mathsf{e}^{-x}, \qquad \Phi(y) = \Upsilon(x)\,\left(\frac{\mathsf{d}x}{\mathsf{d}y}\right)^{-\frac{1}{2}} \tag{6.46}$$

$$\mathsf{e}^{t} = -\frac{1}{1+\mathsf{e}^{x}}, \qquad \Theta(t) = \Upsilon(x)\,\left(\frac{\mathsf{d}x}{\mathsf{d}y}\right)^{-\frac{1}{2}}\,, \qquad \bar{\lambda} = \mathrm{i}\,\lambda\,\mathsf{e}^{\frac{1}{2}\mathrm{i}\pi\delta} \tag{6.47}$$

Then equation (6.8) assumes the following forms:

$$\left[-\frac{\mathsf{d}^2}{\mathsf{d}x^2} + p_1^2\,\frac{\mathsf{e}^{x}}{1+\mathsf{e}^{x}} + \left(\tfrac{1}{4}-p_2^2\right)\,\frac{\mathsf{e}^{x}}{(1+\mathsf{e}^{x})^2} + \lambda^2\,\left(1+\mathsf{e}^{x}\right)^{\delta-1}\right]\Upsilon(x) = 0\,. \tag{6.48}$$

$$\left[-\frac{\mathsf{d}^2}{\mathsf{d}y^2}+\frac{p_2^2}{1+\mathsf{e}^y}+\left(\tfrac{1}{4}-p_1^2\right)\frac{\mathsf{e}^y}{(1+\mathsf{e}^y)^2}+\lambda^2\left(1+\mathsf{e}^{-y}\right)^{-1-\delta}\right]\Phi(y)=0\,. \tag{6.49}$$

$$\left[-\frac{\mathsf{d}^2}{\mathsf{d}t^2}+p_2^2\,\frac{\mathsf{e}^t}{1+\mathsf{e}^t}+\frac{p_1^2}{(1+\mathsf{e}^t)}+\frac{\mathsf{e}^t}{4(1+\mathsf{e}^t)^2}+\bar{\lambda}^2\,\mathsf{e}^{(1-\delta)t}\left(1+\mathsf{e}^t\right)^{-2}\right]\Theta(t)=0 \tag{6.50}$$

We want to construct analogues of solutions $\varphi^{(i)}$ in these coordinates, which were defined as eigenfunction of transformations $\widehat{\Omega}_i$. In coordinate x their action reads:

$$\begin{array}{llllll} \widehat{\Omega}_1: & x & \to & x-2\pi\mathrm{i}\,, & \lambda\to\mathsf{e}^{\mathrm{i}\pi\delta}\lambda & \text{for}\quad x>0 \\ \widehat{\Omega}_3: & x & \to & x+2\pi\mathrm{i}\,, & \lambda\to\lambda & \text{for}\quad x<0\,. \end{array}$$

Coordinate x in the definition above is assumed to be real. The substitution of coordinate is in fact an analytical continuation along the straight line parallel to the imaginary axis. The action of transformations $\widehat{\Omega}_1$, $\widehat{\Omega}_3$ can be defined for x outside of specified ranges. Such definition, in accordance with general definition (6.29), would involve analytical continuation along the deformed contour which crosses or avoids (for the case of $\widehat{\Omega}_3$) same branch cut (see discussion below). Similarly, action of relevant transformations can be specified for (6.49), (6.50):

$$\begin{array}{llllll} \widehat{\Omega}_2: & y & \to & y+2\pi\mathrm{i}\,, & \lambda\to\mathsf{e}^{-\mathrm{i}\pi\delta}\lambda & \text{for}\quad y<0 \\ \widehat{\Omega}_3: & y & \to & y-2\pi\mathrm{i}\,, & \lambda\to\lambda & \text{for}\quad y>0\,. \\ \widehat{\Omega}_1: & t & \to & t+2\pi\mathrm{i}\,, & \bar{\lambda}\to\mathsf{e}^{\mathrm{i}\pi\delta}\bar{\lambda} & \text{for}\quad t<0 \\ \widehat{\Omega}_2: & t & \to & t-2\pi\mathrm{i}\,, & \bar{\lambda}\to\mathsf{e}^{-\mathrm{i}\pi\delta}\bar{\lambda} & \text{for}\quad t>0\,. \end{array}$$

For each coordinate we have specified only two transformations out of three. The third one is best implemented by making use of identity (6.43).

It is necessary to say few words about branch cuts here. Since r.h.s. of (6.2) is not a single-valued function, it is necessary to select a particular branch and draw branch cuts in order to define equation (6.8) properly. Let us assume that

$$\mathcal{P}_{mv}(z)=\frac{\mathcal{P}(z)}{(\zeta-\frac{z_2-z_3}{z_2-z_1})^4}\,\frac{(z_3-z_1)^2(z_2-z_3)^2}{(z_2-z_1))^2} \tag{6.51}$$

is real whenever

$$\zeta=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\qquad \zeta-\mathrm{i}0\in(0;1) \tag{6.52}$$

and branch cut connects points z_1 and z_2. It is convenient to parametrise it by ζ in the following way

$$\zeta\in(0;1) \tag{6.53}$$

Then in the rest of complex plane function $\mathcal{P}(z)$ is defined by analytical continuation.

Such choice leads to (6.48) with branch cut

$$x = \mathrm{i}\pi + t \quad t \in (0; +\infty) \tag{6.54}$$

and $(1+\mathsf{e}^x)^\delta$ in potential taking real values on real axis. To relate solutions of (6.8) to solutions of (6.48) we also have to fix the branch of $\sqrt{\mathsf{d}x/\mathsf{d}z}$. We draw branch cut as

$$\zeta \in (0; +\infty) \tag{6.55}$$

The choice of sign is unimportant as long as it is applied consistently to all solutions. We introduce three pairs of solutions

$$\begin{array}{llll} \Upsilon^{(1)}_{\pm}(x) \sim \mathsf{e}^{\mp p_1 x} & \text{as} \quad x \to +\infty\,, & \Upsilon^{(3)}_{\pm}(x) \sim \mathsf{e}^{\pm\mu x} & \text{as} \quad x \to -\infty \\ \Phi^{(2)}_{\pm}(y) \sim \mathsf{e}^{\pm p_2 y} & \text{as} \quad y \to -\infty\,, & \Phi^{(3)}_{\pm}(y) \sim \mathsf{e}^{\mp\mu y} & \text{as} \quad y \to +\infty \\ \Theta^{(1)}_{\pm}(t) \sim \mathsf{e}^{\pm p_1 t} & \text{as} \quad t \to -\infty\,, & \Theta^{(2)}_{\pm}(t) \sim \mathsf{e}^{\mp p_2 t} & \text{as} \quad t \to +\infty \end{array} \tag{6.56}$$

that are eigenfunctions of transformations $\widehat{\Omega}_1$, $\widehat{\Omega}_2$, $\widehat{\Omega}_3$ respectively

$$\begin{array}{ll} \widehat{\Omega}_1\big(\Upsilon^{(1)}_{\pm}\big) = \mathsf{e}^{\pm 2\pi \mathrm{i} p_1}\,\Upsilon^{(1)}_{\pm}\,, & \widehat{\Omega}_3\big(\Upsilon^{(3)}_{\pm}\big) = \mathsf{e}^{\pm 2\pi \mathrm{i}\lambda}\,\Upsilon^{(3)}_{\pm}\,, \\ \widehat{\Omega}_1\big(\Theta^{(1)}_{\pm}\big) = \mathsf{e}^{\pm 2\pi \mathrm{i} p_1}\,\Theta^{(1)}_{\pm}\,, & \widehat{\Omega}_2\big(\Theta^{(2)}_{\pm}\big) = \mathsf{e}^{\pm 2\pi \mathrm{i} p_2}\,\Theta^{(2)}_{\pm}\,, \\ \widehat{\Omega}_2\big(\Phi^{(2)}_{\pm}\big) = \mathsf{e}^{\pm 2\pi \mathrm{i} p_2}\,\Phi^{(2)}_{\pm}\,, & \widehat{\Omega}_3\big(\Phi^{(3)}_{\pm}\big) = \mathsf{e}^{\pm 2\pi \mathrm{i}\lambda}\,\Phi^{(3)}_{\pm}\,, \end{array} \tag{6.57}$$

and define functions $\mathtt{W}^{(1)}_{\pm}(\lambda)$, $\mathtt{W}^{(2)}_{\pm}(\lambda)$, $\mathtt{W}^{(3)}_{\sigma',\sigma}(\bar\lambda)$ as their Wronskians:

$$\mathtt{W}^{(2)}_{\pm}(\lambda) = \mathrm{Wr}\,[\,\Upsilon^{(1)}_{\pm}, \Upsilon^{(3)}_{+}\,]\,, \tag{6.58}$$

$$\mathtt{W}^{(3)}_{\sigma',\sigma}(\bar\lambda) = \mathrm{Wr}\,[\Theta^{(2)}_{\sigma'}, \Theta^{(1)}_{\sigma}]\,, \qquad (\sigma, \sigma' = \pm)\ . \tag{6.59}$$

$$\mathtt{W}^{(1)}_{\pm}(\lambda) = \mathrm{Wr}\,[\,\Phi^{(3)}_{+}, \Phi^{(1)}_{\pm}\,]\,, \tag{6.60}$$

Note that if we make the following substitutions in (6.49)

$$p_1 \leftrightarrow p_2\,, \qquad \delta \to -\delta \tag{6.61}$$

followed by change of variables

$$y \to -x\,, \qquad \Phi(y) \to \mathrm{i}\Upsilon(x) \tag{6.62}$$

we shall get (6.48). Consequently, the following identity holds:

$$\mathtt{W}^{(1)}_{\pm}(\lambda \,|\, p_1, p_2, \delta) = \mathtt{W}^{(2)}_{\pm}(\lambda \,|\, p_2, p_1, -\delta) \tag{6.63}$$

Solutions Υ_σ, Φ_σ and Θ_σ are related to solutions $\varphi^{(i)}_\sigma(z)$ in the following way

$$\begin{array}{lllllll}\Upsilon^{(1)}_\sigma(x) & \mapsto & \mathrm{i}\sqrt{2p_1}\mathsf{e}^{-\mathrm{i}\sigma\pi p_1}\hat{\mathfrak{S}}_1^{-\sigma}\varphi^{(1)}_\sigma(z), & \Upsilon^{(3)}_{\sigma''}(x) & = & \sqrt{2\lambda}\varphi^{(3)}_{\sigma''}(z) \\ \Phi^{(2)}_{\sigma'}(y) & \mapsto & \mathrm{i}\sqrt{2p_2}\hat{\mathfrak{S}}_2^{-\sigma'}\varphi^{(2)}_{\sigma'}(z), & \Phi^{(3)}_{\sigma''}(y) & \mapsto & \sqrt{2\lambda}\mathsf{e}^{-\mathrm{i}\sigma''\pi\lambda}\varphi^{(3)}_{\sigma''}(z)\end{array} \tag{6.64}$$

$$\begin{array}{lll}\Theta^{(1)}_\sigma(t;\bar\lambda) & \mapsto & \sqrt{2p_1}\hat{\mathfrak{S}}_1^{-\sigma}\varphi^{(1)}_\sigma(z;\lambda) \\ \Theta^{(2)}_{\sigma'}(t;\bar\lambda) & \mapsto & \mathrm{i}\sqrt{2p_2}\mathsf{e}^{-\mathrm{i}\sigma'\pi p_2}\hat{\mathfrak{S}}_2^{-\sigma}\varphi^{(2)}_{\sigma'}(z;\lambda)\end{array} \tag{6.65}$$

where constants $\hat{\mathfrak{S}}$ hold dependence on z_i:

$$\hat{\mathfrak{S}}_1 = \frac{1}{\sqrt{2p_1}}\left|\frac{z_{23}}{z_{21}z_{13}}\right|^{-p_1}\mathsf{e}^{\frac{\eta^{(1)}_{reg}}{2}}, \qquad \hat{\mathfrak{S}}_2 = \frac{1}{\sqrt{2p_2}}\left|\frac{z_{31}}{z_{23}z_{12}}\right|^{-p_2}\mathsf{e}^{\frac{\eta^{(2)}_{reg}}{2}} \tag{6.66}$$

Then connection matrices $\boldsymbol{S}$ can be expressed in terms of unambiguously defined functions $\mathtt{W}^{(i)}$:

$$\sigma''\boldsymbol{S}^{(3,1)}_{-\sigma'',\sigma}(\lambda) = \frac{\mathrm{i}\mathsf{e}^{\mathrm{i}\sigma\pi p_1}}{2\sqrt{p_1\lambda}}\hat{\mathfrak{S}}_1^{\sigma}\mathtt{W}^{(2)}_\sigma(\sigma''\lambda) \tag{6.67}$$

$$\sigma'\boldsymbol{S}^{(2,3)}_{-\sigma',\sigma''}(\lambda) = \frac{\mathrm{i}\mathsf{e}^{\mathrm{i}\sigma''\pi\lambda}}{2\sqrt{p_2\lambda}}\hat{\mathfrak{S}}_2^{\sigma'}\mathtt{W}^{(1)}_{\sigma'}(\sigma''\lambda) \tag{6.68}$$

$$\sigma\boldsymbol{S}^{(1,2)}_{-\sigma,\sigma'}(\lambda) = \frac{\mathrm{i}\mathsf{e}^{\sigma'\mathrm{i}\pi p_2}}{2\sqrt{p_1p_2}}\hat{\mathfrak{S}}_1^{\sigma}\hat{\mathfrak{S}}_2^{\sigma'}\mathtt{W}^{(3)}_{\sigma',\sigma}(\mathrm{i}\lambda\mathsf{e}^{\frac{\mathrm{i}\pi\delta}{2}}) \tag{6.69}$$

6.3 ODE/IM Correspondence for massless Bukhvostov-Lipatov model

From general functional relation (6.44) and relations (6.67) the following functional relations can be derived (see appendix A for details):

$$\begin{aligned}\sin(2\pi\lambda)\,\mathtt{W}^{(1)}_\sigma(-\lambda)\mathtt{W}^{(2)}_{\sigma'}(\lambda) &= \lambda\,\mathsf{e}^{\mathrm{i}\pi(-\sigma'p_1+\sigma p_2-\lambda)}\,\mathtt{W}^{(3)}_{\sigma\sigma'}(\mathrm{i}\lambda q) \\ &+\lambda\,\mathsf{e}^{-\mathrm{i}\pi(-\sigma'p_1+\sigma p_2-\lambda)}\,\mathtt{W}^{(3)}_{\sigma\sigma'}(\mathrm{i}\lambda/q)\end{aligned} \tag{6.70}$$

$$\begin{aligned}2p_1\,\sin(2\pi\lambda)\,\mathtt{W}^{(1)}_\sigma(\lambda)\mathtt{W}^{(1)}_\sigma(-\lambda) &= \mathrm{i}\lambda\mathsf{e}^{-2\pi\mathrm{i}p_1}\mathtt{W}^{(3)}_{\sigma+}(\mathrm{i}\lambda q)\mathtt{W}^{(3)}_{\sigma-}(\mathrm{i}\lambda/q) \\ &-\mathrm{i}\lambda\mathsf{e}^{2\pi\mathrm{i}p_1}\mathtt{W}^{(3)}_{\sigma+}(\mathrm{i}\lambda/q)\mathtt{W}^{(3)}_{\sigma-}(\mathrm{i}\lambda q)\end{aligned} \tag{6.71}$$

$$\begin{aligned}2p_2\,\sin(2\pi\lambda)\,\mathtt{W}^{(2)}_\sigma(\lambda)\mathtt{W}^{(2)}_\sigma(-\lambda) &= \mathrm{i}\lambda\mathsf{e}^{-2\pi\mathrm{i}p_2}\mathtt{W}^{(3)}_{+\sigma}(\mathrm{i}\lambda/q)\mathtt{W}^{(3)}_{-\sigma}(\mathrm{i}\lambda q) \\ &-\mathrm{i}\lambda\mathsf{e}^{2\pi\mathrm{i}p_2}\mathtt{W}^{(3)}_{+\sigma}(\mathrm{i}\lambda q)\mathtt{W}^{(3)}_{-\sigma}(\mathrm{i}\lambda/q)\end{aligned} \tag{6.72}$$

$$\begin{aligned} 2\lambda \mathtt{T}^{(3)}\mathtt{W}^{(3)}_{\sigma'\sigma}(\mathrm{i}\lambda) &= \mathsf{e}^{-\mathrm{i}\pi(\sigma p_1+\sigma' p_2)+\mathrm{i}\pi\lambda\cos\left(\frac{\pi\delta}{2}\right)}\mathtt{W}^{(1)}_{\sigma'}(\lambda/q)\mathtt{W}^{(2)}_{\sigma}(\lambda q) \\ &+\mathsf{e}^{\mathrm{i}\pi(\sigma p_1+\sigma' p_2)-\mathrm{i}\pi\lambda\cos\left(\frac{\pi\delta}{2}\right)}\mathtt{W}^{(1)}_{\sigma'}(\lambda q)\mathtt{W}^{(2)}_{\sigma}(\lambda/q) \end{aligned} \tag{6.73}$$

where $q = \mathsf{e}^{\mathrm{i}\frac{\pi\delta}{2}}$ and

$$\begin{aligned} 2\lambda \mathtt{T}^{(3)}(\lambda) =& \frac{\mathsf{e}^{-\frac{\mathrm{i}\pi}{2}\lambda(q-q^{-1})}}{2\mathrm{i}p_1}\left(\mathsf{e}^{2\pi\mathrm{i}p_1}\mathtt{W}^{(2)}_{+}(\lambda q)\mathtt{W}^{(2)}_{-}(\lambda/q) - \mathsf{e}^{-2\mathrm{i}\pi p_1}\mathtt{W}^{(2)}_{+}(\lambda/q)\mathtt{W}^{(2)}_{-}(\lambda q)\right) = \\ &-\frac{\mathsf{e}^{\frac{\mathrm{i}\pi}{2}\lambda(q-q^{-1})}}{2\mathrm{i}p_2}\left(\mathsf{e}^{-2\pi\mathrm{i}p_2}\mathtt{W}^{(1)}_{+}(\lambda q)\mathtt{W}^{(1)}_{-}(\lambda/q) - \mathsf{e}^{2\mathrm{i}\pi p_2}\mathtt{W}^{(1)}_{+}(\lambda/q)\mathtt{W}^{(1)}_{-}(\lambda q)\right) \end{aligned} \tag{6.74}$$

Extra relations follow trivially from properties (6.41) of the connection matrices:

$$\mathtt{W}^{(1)}_{+}(\lambda)\mathtt{W}^{(1)}_{-}(-\lambda) - \mathtt{W}^{((1)}_{+}(-\lambda)\mathtt{W}^{(1)}_{-}(\lambda) \;=\; -4p_2\lambda \tag{6.75}$$

$$\mathtt{W}^{(2)}_{+}(\lambda)\mathtt{W}^{(2)}_{-}(-\lambda) - \mathtt{W}^{(2)}_{+}(-\lambda)\mathtt{W}^{(2)}_{-}(\lambda) \;=\; -4p_1\lambda \tag{6.76}$$

Functional relations restrict positions of zeroes of functions $\mathtt{W}^{(i)}$. Indeed, let $-\lambda^{(1)}$ be a zero of $\mathtt{W}^{(1)}_{\sigma}$. Then left hand side of (6.70) vanishes and we obtain the following equation:

$$-1 \;=\; \mathsf{e}^{2\pi\mathrm{i}(\sigma' p_1-\sigma p_2+\lambda^{(1)})}\frac{\mathtt{W}^{(3)}_{\sigma\sigma'}(\mathrm{i}\lambda^{(1)}/q)}{\mathtt{W}^{(3)}_{\sigma\sigma'}(\mathrm{i}\lambda^{(1)}q)} \tag{6.77}$$

Plugging in the same relation (6.70) argument $\lambda = \lambda^{(3)}q^{\pm1}$ with $\lambda^{(3)}$ satisfying condition $\mathtt{W}^{(3)}_{\sigma'\sigma}(\mathrm{i}\lambda^{(3)}) = 0$, we find

$$\sin\left(2\pi\lambda^{(3)}q^{\pm1}\right)\mathtt{W}^{(1)}_{\sigma'}(-\lambda^{(3)}q^{\pm1})\mathtt{W}^{(2)}_{\sigma}(\lambda^{(3)}q^{\pm1}) = \lambda\mathsf{e}^{\pm\mathrm{i}\pi(-\sigma p_1+\sigma' p_2-\lambda q^{\pm1})}\mathtt{W}^{(3)}_{\sigma'\sigma}(\mathrm{i}\lambda^{(3)}q^{\pm2}) \tag{6.78}$$

Setting $\lambda = \lambda^{(3)}$ in (6.73) we obtain yet another equation

$$-1 \;=\; \mathsf{e}^{-2\mathrm{i}\pi(\sigma p_1+\sigma' p_2)+\mathrm{i}\pi\lambda^{(3)}(q+q^{-1})}\frac{\mathtt{W}^{(1)}_{\sigma'}(\lambda^{(3)}q)\mathtt{W}^{(2)}_{\sigma}(\lambda^{(3)}/q)}{\mathtt{W}^{(1)}_{\sigma'}(\lambda^{(3)}/q)\mathtt{W}^{(2)}_{\sigma}(\lambda^{(3)}q)} \tag{6.79}$$

Now expressing $\mathtt{W}^{(2)}$ in (6.79) via (6.78) and using the properties of gamma functions

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin\pi x}\,, \qquad \Gamma(x+1) = x\Gamma(x) \tag{6.80}$$

we get equation

$$-1 \;=\; \mathsf{e}^{-4\mathrm{i}\pi\sigma p_1}\frac{\mathtt{W}^{(1)}_{\sigma'}(\lambda^{(3)}q)\mathtt{W}^{(1)}_{\sigma'}(-\lambda^{(3)}q)\mathtt{W}^{(3)}_{\sigma',\sigma}(\mathrm{i}\lambda^{(3)}/q^2)\Gamma(1+2\lambda^{(3)}/q)\Gamma(1-2\lambda^{(3)}/q)}{\mathtt{W}^{(1)}_{\sigma'}(\lambda^{(3)}/q)\mathtt{W}^{(1)}_{\sigma'}(-\lambda^{(3)}/q)\mathtt{W}^{(3)}_{\sigma',\sigma}(\mathrm{i}\lambda^{(3)}q^2)\Gamma(1+2\lambda^{(3)}q)\Gamma(1-2\lambda^{(3)}q)} \tag{6.81}$$

We notice then that making formal substitutions

$$\begin{gathered}\mathtt{W}^{(3)}_{++}(\mathrm{i}\lambda)\to\prod_{\ell}\sinh(\theta-u_{\ell})\qquad \mathtt{W}^{(1)}_{+}(\lambda)/\Gamma(1+2\lambda)\to\prod_{\jmath}\sinh\left(\tfrac{1}{2}(\theta-\theta_{\jmath})\right)\\ \lambda^{(1)}\to-\mathsf{e}^{\theta_{\jmath}}\qquad \lambda^{(3)}\to\mathsf{e}^{u_{\ell}}\end{gathered}\tag{6.82}$$

in equations (6.77), (6.81) we obtain Bethe Ansatz equations for Bukhvostov-Lipatov model (5.33), (5.31). This is an argument in favour of the ODE/IM correspondence.

Note, however, that the number of roots in right hand side of (6.82) is not specified. In fact functions $\mathtt{W}^{(i)}$ have infinite number of zeroes as it will be made clear by quasiclassical analysis. To complete the proof of the correspondence it remains to demonstrate that zeroes of functions $\mathtt{W}^{(i)}$ (or their massive analogues) follow the same phase assignment as Bethe roots, and that roots of regularized Bethe equations converge pointwise to zeroes of connection coefficients. In chapter 7 we shall prove that equations (6.77), (6.81) are exact equations describing vacuum state of Bukhvostov-Lipatov model.

6.4 Auxiliary linear problem. Massive case

Now we can generalise above construction to massive case. Consider full matrix auxiliary linear problem (6.5). Firstly, we prove that its solutions still form a two dimensional linear space. Indeed, let us assume that $\boldsymbol{\Psi}_+$, $\boldsymbol{\Psi}_-$ are two linearly independent solutions of (6.6), and no component of either solution is identically zero. Then both admit representations (6.7) and (6.9). From (6.8),(6.10) it follows that

$$\partial_z \mathrm{Wr}\,[\psi_+,\psi_-]=0\qquad \partial_{\bar z}\overline{\mathrm{Wr}}\,[\bar\psi_+,\bar\psi_-]=0\tag{6.83}$$

Then, from (6.11) we have

$$\det(\boldsymbol{\Psi}_+,\boldsymbol{\Psi}_-)(z,\bar z)=\mathrm{Wr}\,[\psi_+,\psi_-](\bar z)=\overline{\mathrm{Wr}}\,[\bar\psi_+,\bar\psi_-](z)\tag{6.84}$$

where $(\boldsymbol{\Psi}_+,\boldsymbol{\Psi}_-)$ is a 2×2 matrix with columns $\boldsymbol{\Psi}_+$,$\boldsymbol{\Psi}_-$ and derivatives in rightmost Wronskian are computed w.r.t. $\bar z$. It is only possible if all three expressions are constants. Obviously, if solutions were linearly dependent determinant would be identically zero. Conversely, if determinant is identically zero solutions must be linearly dependent. So if there are three linearly independent solutions, there exists non-zero complex numbers c_2, c_3 such that

$$\det(\boldsymbol{\Psi}_1,c_2\boldsymbol{\Psi}_2)=1\qquad \det(\boldsymbol{\Psi}_1,c_3\boldsymbol{\Psi}_3)=1\tag{6.85}$$

Then, since determinant is linear, we arrive to contradiction:

$$\det(\mathbf{\Psi}_1, c_2\mathbf{\Psi}_2 - c_3\mathbf{\Psi}_3) = 0 \tag{6.86}$$

The existence of nontrivial solutions such that one of its components is identically zero can be ruled out by plugging this ansatz into (6.6) and trying to solve perturbatively near one of the singularities. So, even though the auxiliary linear problem is now a system of PDEs, its solutions still form two dimensional linear space. So we can, just as in conformal case, introduce three bases of solutions defined as eigenfunctions of $\widehat{\Omega}_i$. We write each of them in the form of fundamental matrix:

$$\mathbf{\Psi}^{(i)} = \left(\mathbf{\Psi}_-^{(i)}, \mathbf{\Psi}_+^{(i)}\right) \in \mathbb{SL}(2,\mathbb{C}) \qquad (i = 1,2,3) \tag{6.87}$$

Let the solution $\eta(z)$ of mShG satisfy same asymptotic conditions (6.22,6.23). In this case existence and uniqueness were proven rigorously only for $p_1 = p_2 = 0$ (see p.e. the argument in [74]). However, there is no reason to doubt that it is the case for sufficiently small p_i. Solving (6.6) perturbatively in $z - z_i$ in next to leading order we obtain the following asymptotical expressions for solutions $\mathbf{\Psi}^{(i)}$:

$$\mathbf{\Psi}^{(i)} \quad \to \quad \left(2p_i\right)^{-\frac{1}{2}\sigma_3}\, \mathsf{e}^{\mathrm{i}\beta_i\sigma_3} \left(\frac{z - z_i}{\bar{z} - \bar{z}_i}\right)^{\frac{1}{4}(1-2p_i)\,\sigma_3} \qquad \text{as} \quad z \to z_i \quad (i = 1,2) \tag{6.88}$$

$$\mathbf{\Psi}_\pm^{(3)} \to \frac{1}{\sqrt{2\lambda}} \left(\frac{z_{13}z_{32}}{z_{12}}\right)^{\mp\rho\lambda} \left(\frac{\bar{z}_{13}\bar{z}_{32}}{\bar{z}_{12}}\right)^{\mp\rho\lambda^{-1}} \begin{pmatrix} (z - z_3)^{+\frac{1}{4}\pm\rho\lambda}\, (\bar{z} - \bar{z}_3)^{-\frac{1}{4}\pm\rho\lambda^{-1}} \\ \pm\,\lambda\,(z - z_3)^{-\frac{1}{4}\pm\rho\lambda}\, (\bar{z} - \bar{z}_3)^{+\frac{1}{4}\pm\rho\lambda^{-1}} \end{pmatrix} \tag{6.89}$$

for $z \to z_3$. Note that notation λ is used in **different** sense than in sections (6.2), (6.3):

$$\lambda = \mathsf{e}^{\theta} \tag{6.90}$$

Unlike Liouville equation (6.13) and corresponding auxiliary linear problem (6.19) general problem (6.6) is no longer invariant under conformal transformations. Therefore it is no longer possible to absorb dependence of connection coefficients on coordinates z_1, z_2, z_3 into a normalization constant like it was done in (6.67). However, as we shall see, it is possible to choose normalization in such way that connection coefficients depend only on absolute values of differences $|z_i - z_j|$. In order to do this we have introduced normalization constants β_i into definitions (6.88):

$$\mathsf{e}^{\mathrm{i}\beta_i} = \left(\frac{z_{ji}z_{ik}}{z_{jk}}\frac{\bar{z}_{jk}}{\bar{z}_{ji}\bar{z}_{ik}}\right)^{\frac{p_i}{2}} \qquad\qquad (z_{ij} = z_i - z_j)\,. \tag{6.91}$$

Parameters are assumed to satisfy the following constraints:

$$\left|\mathrm{Re}\big(\rho\,(\lambda-\lambda^{-1})\big)\right| < \frac{1}{2}\,, \qquad 0 < p_i < \frac{a_i}{4} \qquad (i=1,2)\,. \tag{6.92}$$

Similarly to conformal case (6.37) we introduce connection matrices relating different bases of solutions

$$\boldsymbol{S}^{(i,j)}_{\sigma,\sigma'}(\lambda) = \det\Big(\boldsymbol{\Psi}^{(j)}_{\sigma'}\,,\ \boldsymbol{\Psi}^{(i)}_{-\sigma}\Big)\,. \tag{6.93}$$

The action of transformations is almost identical to (6.30), (6.31):

$$\widehat{\Omega}_i\big(\boldsymbol{\Psi}^{(i)}\big) \;=\; -\boldsymbol{\Psi}^{(i)}\,\mathsf{e}^{-2\pi\mathrm{i}p_i(\lambda)\sigma_3} \tag{6.94a}$$

$$\widehat{\Omega}_j\big(\boldsymbol{\Psi}^{(i)}\big) \;=\; -\boldsymbol{\Psi}^{(i)}\,\boldsymbol{S}^{(i,j)}(\lambda)\,\mathsf{e}^{-2\pi\mathrm{i}p_j(\lambda)\sigma_3}\,\boldsymbol{S}^{(j,i)}(\lambda\,q_j^{-1}) \tag{6.94b}$$

$$\widehat{\Omega}_k\big(\boldsymbol{\Psi}^{(i)}\big) \;=\; -\boldsymbol{\Psi}^{(i)}\,\boldsymbol{S}^{(i,k)}(\lambda)\,\mathsf{e}^{-2\pi\mathrm{i}p_k(\lambda)\sigma_3}\,\boldsymbol{S}^{(k,i)}(\lambda\,q_k^{-1})\,, \tag{6.94c}$$

where we have introduced shortcut notation $p_3(\lambda)$ allowing to write it down as uniform expression:

$$p_1(\lambda)\equiv p_1, \qquad p_2(\lambda)\equiv p_2, \qquad p_3(\lambda) = \rho\,(\lambda-\lambda^{-1})\,. \tag{6.95}$$

Then we can repeat all the reasoning leading to functional relations (6.44), (6.41). Indeed, for (i,j,k) being a cyclic permutation of $(1,2,3)$ we must have

$$\widehat{\Omega}_k\circ\widehat{\Omega}_j\circ\widehat{\Omega}_i\,\big(\boldsymbol{\Psi}^{(i)}\big) = \boldsymbol{\Psi}^{(i)}\,. \tag{6.96}$$

It is convenient to introduce an analogue to gamma function for massive case:

$$\begin{aligned}
\mathsf{Z}(\lambda)\Big|_{\lambda=\mathrm{e}^{\theta}} \;&=\; \frac{\big(2\rho\;\mathsf{e}^{\gamma_E-1}\big)^{-2\rho\cosh(\theta)}}{4\sqrt{\pi\rho}\,\cosh(\frac{\theta}{2})}\prod_{n=1}^{\infty}\left(\sqrt{1+\Big(\frac{4\rho}{n}\Big)^2}+\frac{4\rho\cosh(\theta)}{n}\right)^{-1}\mathsf{e}^{\frac{4\rho}{n}\cosh(\theta)}\\
&=\; \exp\left(4\rho\,\theta\sinh\theta-\int_{-\infty}^{\infty}\frac{\mathsf{d}\tau}{2\pi}\,\frac{\log(1-\mathsf{e}^{-8\pi\rho\cosh(\tau)})}{\cosh(\theta-\tau)}\right)\,.
\end{aligned} \tag{6.97}$$

Then connection matrices can be parametrized in terms of new functions $\mathsf{W}^{(i)}$ which are, by construction, entire functions

$$
\begin{aligned}
\boldsymbol{S}^{(2,3)}_{-\sigma',\sigma''}(\lambda) &= -\frac{\mathrm{i}\sigma'}{\sqrt{2}}\,\mathsf{W}^{(1)}_{\sigma'}(\sigma''\,\lambda)\ \mathsf{Z}(\sigma''\,\lambda)\,\mathsf{e}^{\mathrm{i}\pi\sigma''\rho(\lambda-\lambda^{-1})} \\
\boldsymbol{S}^{(3,1)}_{-\sigma'',\sigma}(\lambda) &= -\frac{\mathrm{i}\sigma''}{\sqrt{2}}\,\mathsf{W}^{(2)}_{\sigma}(\sigma''\,\lambda)\ \mathsf{Z}(\sigma''\,\lambda)\,\mathsf{e}^{\mathrm{i}\pi\sigma p_1} \\
\boldsymbol{S}^{(1,2)}_{-\sigma,\sigma'}(\lambda) &= \frac{\mathrm{i}\sigma}{2}\,\mathsf{W}^{(3)}_{\sigma',\sigma}(\lambda\,\mathsf{e}^{\frac{\mathrm{i}\pi\delta}{2}})\ \mathsf{e}^{\mathrm{i}\pi\sigma' p_2}\ ,
\end{aligned}
\tag{6.98}
$$

Using general functional relations (6.44), (6.41) and definitions (6.98) we obtain (see appendix A) the following functional relations for functions $\mathsf{W}^{(i)}$:

$$2\,\mathsf{W}^{(1)}_{\sigma}(\lambda)\,\mathsf{W}^{(1)}_{\sigma}(-\lambda) = \mathsf{e}^{2\pi\mathrm{i}p_1}\,\mathsf{W}^{(3)}_{\sigma-}(\lambda q)\,\mathsf{W}^{(3)}_{\sigma+}(\lambda/q) \tag{6.99a}$$

$$\qquad - \mathsf{e}^{-2\pi\mathrm{i}p_1}\,\mathsf{W}^{(3)}_{\sigma+}(\lambda q)\,\mathsf{W}^{(3)}_{\sigma-}(\lambda/q) \tag{6.99b}$$

$$2\,\mathsf{W}^{(2)}_{\sigma}(\lambda)\,\mathsf{W}^{(2)}_{\sigma}(-\lambda) = \mathsf{e}^{2\pi\mathrm{i}p_2}\,\mathsf{W}^{(3)}_{-\sigma}(\lambda q)\,\mathsf{W}^{(3)}_{+\sigma}(\lambda/q) \tag{6.99c}$$

$$\qquad - \mathsf{e}^{-2\pi\mathrm{i}p_2}\,\mathsf{W}^{(3)}_{+\sigma}(\lambda q)\,\mathsf{W}^{(3)}_{-\sigma}(\lambda/q) \tag{6.99d}$$

$$2\,\mathsf{W}^{(1)}_{\sigma'}(\lambda)\,\mathsf{W}^{(2)}_{\sigma}(-\lambda) = \mathsf{e}^{-\mathrm{i}\pi(p_1\sigma-p_2\sigma'-\rho(\lambda-\lambda^{-1}))}\,\mathsf{W}^{(3)}_{\sigma'\sigma}(\lambda q) \tag{6.99e}$$

$$\qquad + \mathsf{e}^{\mathrm{i}\pi(p_1\sigma-p_2\sigma'-\rho(\lambda-\lambda^{-1}))}\,\mathsf{W}^{(3)}_{\sigma'\sigma}(\lambda/q) \tag{6.99f}$$

$$
\begin{aligned}
\mathsf{W}^{(3)}_{\sigma'\sigma}(\lambda)\,\mathsf{T}^{(3)}(\lambda) &= \mathsf{e}^{\mathrm{i}\pi\left(p_1\sigma+p_2\sigma'-\rho(\lambda-\lambda^{-1})\cos(\frac{\pi\delta}{2})\right)}\ \mathsf{W}^{(1)}_{\sigma'}(\lambda q)\mathsf{W}^{(2)}_{\sigma}(\lambda/q) \\
&+ \mathsf{e}^{-\mathrm{i}\pi\left(p_1\sigma+p_2\sigma'-\rho(\lambda-\lambda^{-1})\cos(\frac{\pi\delta}{2})\right)}\ \mathsf{W}^{(1)}_{\sigma'}(\lambda/q)\mathsf{W}^{(2)}_{\sigma}(\lambda q)\big)\,,
\end{aligned}
\tag{6.99g}
$$

where $q = \mathsf{e}^{\mathrm{i}\frac{\pi\delta}{2}}$ and

$$
\begin{aligned}
\mathsf{T}^{(3)}(\lambda) &= \frac{\mathsf{e}^{+\rho(\lambda+\lambda^{-1})\sin(\frac{\pi\delta}{2})}}{2\mathrm{i}}\ \Big(\mathsf{e}^{-2\pi\mathrm{i}p_2}\ \mathsf{W}^{(1)}_{+}(\lambda/q)\,\mathsf{W}^{(1)}_{-}(\lambda q) - \mathsf{e}^{2\pi\mathrm{i}p_2}\ \mathsf{W}^{(1)}_{+}(\lambda q)\,\mathsf{W}^{(1)}_{-}(\lambda/q)\Big) \\
&= \frac{\mathsf{e}^{-\rho(\lambda+\lambda^{-1})\sin(\frac{\pi\delta}{2})}}{2\mathrm{i}}\Big(\mathsf{e}^{-2\pi\mathrm{i}p_1}\ \mathsf{W}^{(2)}_{+}(\lambda q)\,\mathsf{W}^{(2)}_{-}(\lambda/q) - \mathsf{e}^{2\pi\mathrm{i}p_1}\ \mathsf{W}^{(2)}_{+}(\lambda/q)\,\mathsf{W}^{(2)}_{-}(\lambda q)\Big)\,.
\end{aligned}
\tag{6.99h}
$$

6.5 Pattern of zeroes of connection coefficients

The Bethe Ansatz equations alone are not a sufficient input for the problem of finding the set of Bethe roots describing the physical vacuum state. In order to find some solution one has to make an assumption regarding the asymptotical behaviour of the involved functions, which can in principle affect the answer. In the QFT framework it is absolutely not clear which choice is preferred. ODE/IM correspondence provides us with some intuition. Indeed, connection coefficients of an ODE are unambiguously defined quantities, and so are their asymptotic expansions. The conjecture central to this thesis is that zeroes of Wronskians coincide with Bethe roots. In other words, ODE/IM correspondence automatically gives us correct prescription for large λ asymptotics of functions $\mathtt{W}$ and W. In this section we address the problem of determining the pattern of those zeroes. We shall concentrate on conformal case as numerical solution in this case is much simpler, but as we shall see the massive case will require only slight modifications. In chapter 7 we will reduce the system of Bethe equations to a finite system of non-linear integral equations. This derivation requires to know the position of zeroes and singularities of functions $\mathtt{W}^{(i)}$ with precision sufficient to tell on which side of the contour of integration they fall. In this section we shall use quasiclassical approximation for large λ and numerical methods for small λ.

6.5.1 Bethe roots for free fermions

To obtain some intuition about behaviour of zeroes and to ensure consistency of ODE/IM approach it it instructive to consider case of free fermions ($\delta = 0$). The ODE can be written in the following form suitable for computing $\mathtt{W}^{(2)}$:

$$-y''(z) + V(z)y(z) = 0 \tag{6.100}$$

with potential

$$V(z) = \frac{\lambda^2}{(1-z)^2 z^{2-a_1}} + \frac{p_1^2 - \frac{1}{4}}{z^2(1-z)} - \frac{1}{4z(1-z)^2} + \frac{\frac{1}{4} - p_2^2}{z(1-z)} \tag{6.101}$$

At $\delta = 0$ this ODE can be solved explicitly in terms of hypergeometric functions. Consider solutions $\psi^{(1)}_{\sigma}$ and $\psi^{(3)}_{\sigma''}$ defined by

$$\psi^{(1)}_{\sigma} = z^{\frac{1}{2}+\sigma p_1}(1-z)^{\frac{1}{2}+\lambda}{}_2F_1(\tfrac{1}{2}+\sigma p_1+p_2+\lambda, \tfrac{1}{2}+\sigma p_1-p_2+\lambda, 1+2\sigma p_1; z) \tag{6.102}$$

and

$$\psi^{(3)}_{\sigma''} = z^{\frac{1}{2}+p_1}(1-z)^{\frac{1}{2}+\sigma''\lambda}{}_2F_1(\tfrac{1}{2}+p_1+p_2+\sigma''\lambda, \tfrac{1}{2}+p_1-p_2+\sigma''\lambda, 1+2\sigma''\lambda; 1-z) \tag{6.103}$$

Since hypergeometric function ${}_2F_1(a,b,c;z)$ is regular for $|z|<1$ solutions $\psi^{(1)}_{\sigma}$ and $\psi^{(3)}_{\sigma''}$ are eigenfunctions of transformations $\widehat{\Omega}_1$ and $\widehat{\Omega}_3$ respectively, which in this case amount to analytical continuation around points $z=0$ and $z=1$ respectively.

Using linear relations between Kummer's solutions of hypergeometric equation it is straightforward to compute wronskian $\mathtt{W}^{(2)}_\sigma$:

$$\mathtt{W}^{(2)}_\sigma(\lambda)=\frac{\Gamma(1+2\sigma p_1)\Gamma(1+2\lambda)}{\Gamma(\frac{1}{2}+\sigma p_1+p_2+\lambda)\Gamma(\frac{1}{2}+\sigma p_1-p_2+\lambda)} \tag{6.104}$$

We observe that its zeroes are arranged in close pairs with distance between two zeroes in pair equal to $2p_2$ and their position is described by formula

$$\lambda=-\left(n-\frac{1}{2}+\sigma p_1\pm p_2\right) \tag{6.105}$$

That is, of course, in agreement with Dirac vacuum filled by fermions with negative energies and momenta obeying

$$pR=2\pi(n-\tfrac{1}{2}+k_\pm) \tag{6.106}$$

Exact expression for $\mathtt{W}^{(1)}$ can be obtained from particle-hole duality. In order to compute $\mathtt{W}^{(3)}$ we need the following ODE:

$$-\partial_y^2\chi(y)+\left(\frac{p_1^2}{4(1+\mathsf{e}^y)}+\frac{p_2^2\mathsf{e}^y}{4(1+\mathsf{e}^y)}-\frac{\mathsf{e}^{ya_1}\lambda^2}{4(1+\mathsf{e}^y)^2}+\frac{\mathsf{e}^y}{4(1+\mathsf{e}^y)^2}\right)\chi(y)=0 \tag{6.107}$$

For $\delta=0$ we can solve the ODE explicitly and repeating reasoning leading to (6.104) we get

$$\mathtt{W}^{(3)}_{\sigma'\sigma}(\mathrm{i}\lambda)=\frac{\Gamma(1+2\sigma p_1)\Gamma(1+2\sigma' p_2)}{\Gamma(\frac{1}{2}+\sigma p_1+\sigma' p_2+\lambda)\Gamma(\frac{1}{2}+\sigma p_1+\sigma' p_2-\lambda)} \tag{6.108}$$

and the zeroes are given by

$$\lambda=\mathrm{i}\left(n-\frac{1}{2}-\sigma p_1-\sigma' p_2\right) \tag{6.109}$$

6.5.2 Quasiclassical analysis of connection coefficients

In order to simplify quasiclassical analysis it is convenient to apply Schwarz-Christoffel map

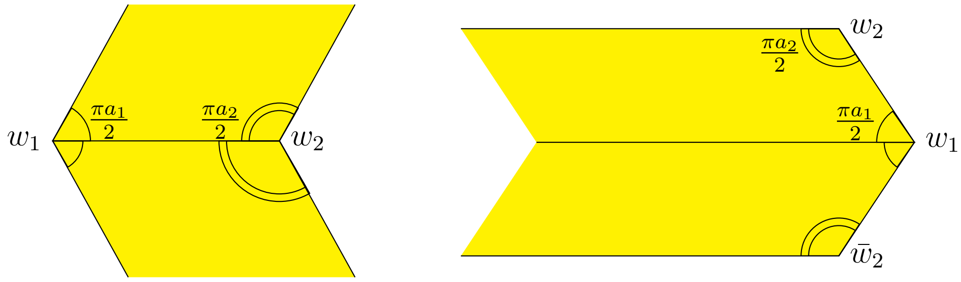
$$w=\rho\int\limits_{z_0}^{z}\sqrt{\mathcal{P}(z)}\mathsf{d}z \tag{6.110}$$

Then PDE (6.1) turns into (non-modified) sinh-Gordon equation

$$\partial_w\partial_{\bar w}\hat\eta=\mathsf{e}^{\hat\eta}+\mathsf{e}^{-\hat\eta} \tag{6.111}$$

with

$$\hat\eta(w)=\eta(z)-\frac{1}{4}\log\left(\mathcal{P}(z)\bar{\mathcal{P}}(\bar z)\right)-\log\rho \tag{6.112}$$



(a) Domain $\mathbb{D}$ for $z_0 \in [0:1]$ (b) Domain $\mathbb{D}$ for $\zeta_0 \in (-\infty:0)$

In conformal case we take $\rho = 1$ in (6.110), and the ODE (6.26) assumes the form of Schrödinger equation with energy $-\lambda^2$

$$-\partial_w^2 \hat{\varphi}(w) + \hat{u}(w)\hat{\varphi}(w) + \lambda^2 \hat{\varphi}(w) = 0 \tag{6.113}$$

If we introduce new coordinate ζ such that z_1, z_2, z_3 map to $0, 1, \infty$ respectively (we can always achieve this by applying a global conformal transformation) and choose $\zeta_0 \in [0:1]$ then image of map (6.110) is two strips (6.1a). The advantage of that choice is that $\zeta = \mathsf{e}^{-x}$, and thus basic solutions can be mapped into solutions (6.56) without extra normalization factors. On the other hand if we take ζ_0 to lie in $(-\infty:0)$ we get two strips glued together along infinite side (6.1b). Similar result, albeit with different orientation, we get if $\zeta_0 \in (1:\infty)$. The potential $\hat{u}$ is now given by

$$\hat{u}(w) = \frac{1}{\mathcal{P}(z)} \left(u_z(z) - \frac{4\mathcal{P}(z)\partial_z^2 \mathcal{P}(z) - 5(\partial_z \mathcal{P}(z))^2}{16\mathcal{P}^2(z)} \right) \tag{6.114}$$

where $\mathcal{P}(z)$ assumes the form

$$\mathcal{P}(z) = \rho^2 z^{a_1 - 2}(1-z)^{a_2 - 2} \tag{6.115}$$

Near punctures potential (6.114) admits the following expansion

$$\hat{u}(w) = \left(\frac{4p_i^2}{a_i^2} - \frac{1}{4} \right) (w - w_i)^{-2} \left(1 + \sum_{n=1}^{\infty} c_n^{(i)} w^{\frac{2n}{a_i}} \right), \qquad w_i = w(z_i)\,, \qquad i = 1, 2 \tag{6.116}$$

For large positive λ there are two turning points located in vicinity of the points w_1, w_2 respectively. They are defined by condition

$$\hat{u}(w_i^*) + \lambda^2 = 0 \tag{6.117}$$

Solving perturbatively we assert that

$$w_i^* = w_i + O(\lambda^{-1}) \tag{6.118}$$

It is not difficult to show that $w_i^*(\lambda)$ are analytical functions of λ^{-1} in some vicinity of point $\lambda^{-1} = 0$. Between these points there is a "forbidden zone". The potential in forbidden zone far from turning points is small but negative, and behaves almost like constant. We define two pairs of quasiclassical solutions on $[w_1 : w_2]$ via the following procedure:

$$\hat{\varphi}_\pm^{(q,i)}(w) = \exp(\pm\lambda|w - w_i| + f_\pm^{(i)}(w)) \tag{6.119}$$

$$f_\pm^{(i)}(w) = \sum_{n=1}^{\infty} \lambda^{-n} f_{n,\pm}^{(i)}(w) \tag{6.120}$$

Derivatives of coefficient functions $f_{n,\pm}^{(i)}$ are λ-independent and can be obtained from the following differential equation

$$-f_\pm'' - (f_\pm')^2 \mp 2\lambda f_\pm + \hat{u}(w) = 0 \tag{6.121}$$

In order to define solution completely one has to specify limits of integration. Integrands in question are divergent at $w = w_i$, and the position of turning point depends on λ introducing fractional powers of λ, so we choose point $w = 0$ (or any other point in the middle of forbidden zone) as reference point.

$$f_\pm^{(i)}(w) = \int_0^w (f_\pm^{(i)}(w'))' \mathsf{d}w' \tag{6.122}$$

For our purposes it is enough to note that

$$\hat{\varphi}_\pm^{(q,i)}(w) = \exp(\pm\lambda|w - w_i| + O(\lambda^{-1})) \tag{6.123}$$

We can drop all terms that vanish at $w = w_1$ from (6.116). Resulting approximate ODE

$$-\partial_w^2\hat{\varphi} + \left(\frac{4p_i^2}{a_i^2} - \frac{1}{4}\right)(w - w_i)^{-2}\hat{\varphi} + \lambda^2\hat{\varphi} = 0 \tag{6.124}$$

can be solved explicitly:

$$\hat{\varphi}_\pm^{(1)}(w) = w^{\frac{1}{2}\pm\frac{2p_1}{a_1}} \mathsf{e}^{\lambda w} {}_1F_1\left(\frac{1}{2} \pm \frac{2p_1}{a_1}, 1 \pm \frac{4p_1}{a_1}; -2\lambda w\right) \tag{6.125}$$

Such solutions are images of solutions $\Theta_\pm^{(1)}$. This approximation is valid for $w \ll 1$. On the other hand, quasiclassical approximation is valid when $w \gg \lambda^{-1}$. For sufficiently large λ there exists an interval where both approximations are valid, and we can compute connection coefficients. In the forbidden zone only decreasing (as function of distance from the turning point) solution is well defined: growing solution is defined up to exponentially small correction. It is possible to find linear combination of solution (6.125) that is exponentially decreasing.

$$\hat{\varphi}_-^{(1,q)} = 2\sqrt{\frac{a_1}{\pi}}\left(-\left(\frac{\lambda}{2}\right)^{\frac{1}{2}+\frac{2p_1}{a_1}}\Gamma\left(1-\frac{2p_1}{a_1}\right)\hat{\varphi}_+^{(1)}+\left(\frac{\lambda}{2}\right)^{\frac{1}{2}-\frac{2p_1}{a_1}}\Gamma\left(1+\frac{2p_1}{a_1}\right)\hat{\varphi}_-^{(1)}\right) \tag{6.126}$$

It follows then that solutions with fixed asymptotics can be decomposed as

$$\hat{\varphi}_\pm^{(1)} = 2\sqrt{\frac{a_1}{\pi}}\left(\frac{\lambda}{2}\right)^{\frac{1}{2}\mp\frac{2p_1}{a_1}}\Gamma\left(1\pm\frac{2p_1}{a_1}\right)\hat{\varphi}_+^{(1,q)}+h_\pm^{(1)}\hat{\varphi}_-^{(1,q)} \tag{6.127}$$

Coefficients $h_\pm^{(1)}$ remain unspecified, but for the purpose of finding large λ asymptotics of Wronskians they are irrelevant. Similar combination can be built near w_2. Then we know coefficients in front of growing quasiclassical solutions in expansion of solutions with defined asymptotics. Now we are ready to compute the Wronskian $\mathtt{W}_{\sigma'\sigma}^{(3)}$

$$\mathrm{Wr}\,[\hat{\varphi}_\sigma^{(2)},\hat{\varphi}_{\sigma'}^{(1)}] \asymp (2\lambda)^{\frac{2\sigma' p_1}{a_1}+\frac{2\sigma p_2}{a_2}}\frac{\sqrt{a_1a_2}}{4\pi\lambda}\Gamma\left(1+\frac{2\sigma' p_1}{a_1}\right)\Gamma\left(1+\frac{2\sigma p_2}{a_2}\right)\mathrm{Wr}\,[\hat{\varphi}_+^{(2,q)},\hat{\varphi}_+^{(1,q)}] \tag{6.128}$$

Writing (6.128) we have omitted corrections to (6.126) originating from subleading terms in (6.116). It is justified as they can be shown to decrease at least as λ^{-1} for large λ. We have also omitted Wronskian $\mathrm{Wr}\,[\hat{\varphi}_-^{(2,q)},\hat{\varphi}_-^{(1,q)}]$ because it is proportional to $\exp(-\lambda|w_2-w_1|)$ and is therefore exponentially small. Simplifying, we get

$$\mathtt{W}_{\sigma'\sigma}^{(3)}(\lambda) = \Gamma\big(1+\tfrac{2\sigma p_1}{a_1}\big)\,\Gamma\big(1+\tfrac{2\sigma' p_2}{a_2}\big)\,\frac{\sqrt{a_1a_2}}{2\pi}\,\left(\frac{\lambda}{a_1}\right)^{-\frac{2\sigma p_1}{a_1}}\,\left(\frac{\lambda}{a_2}\right)^{-\frac{2\sigma' p_2}{a_2}}D_{\sigma'\sigma}^{(3)}(\lambda) \tag{6.129}$$

$$\log\big(D_{\sigma'\sigma}^{(3)}(\lambda)\big) \asymp \lambda\int_0^1\sqrt{\mathcal{P}(z)}\mathsf{d}z + O(\lambda^{-\frac{2}{a_2}}) = \frac{\pi\lambda}{\sin(\frac{\pi a_1}{2})}+O(\lambda^{-\frac{2}{a_2}}) \tag{6.130}$$

Similarly, choosing $\zeta_0<0$ ($\zeta_0>1$) we can compute in leading approximation Wronskians $\mathtt{W}^{(i)}$. For the case $\zeta_0=-1$ the relation between w and ζ near point $\zeta=\infty$ can be written as

$$w(\zeta) = \tilde{w}_3+\log(-\zeta)+O(\zeta^{-1})\,,\qquad \tilde{w}_3 = -\int\limits_{-\infty}^{-1}\left((-\zeta)^{\frac{a_1}{2}-1}(1-\zeta)^{\frac{a_2}{2}-1}+\frac{1}{\zeta}\right)\mathsf{d}\zeta \tag{6.131}$$

Solution $\varphi_\pm^{(3)}(z)$ therefore maps to

$$\hat{\varphi}_\pm^{(3)}(w) \asymp \frac{1}{\sqrt{2\lambda}}\exp\left(\mp\lambda(w-\tilde{w}_3)\right) \tag{6.132}$$

Note that in this case there is only one turning point as near $z = \infty$ (which maps to $w = \infty$) the potential vanishes. As a result of these calculations we get

$$\mathtt{W}_{\pm}^{(i)}(\lambda) = -\Gamma(1 \pm k_j)\sqrt{\frac{a_j\lambda}{\pi}}\left(\frac{\lambda}{a_j}\right)^{\mp k_j} D_{\pm}^{(i)}(\lambda) \qquad (i = 1, 2)\,, \tag{6.133}$$

$$\log D_{\pm}^{(2)}(\lambda) \asymp \lambda \left(\int_{-\infty}^{-1} \left((-\zeta)^{\frac{a_1}{2}-1}(1-\zeta)^{\frac{a_2}{2}-1} + \frac{1}{\zeta} \right) \mathsf{d}\zeta + \int_{-1}^{0} (-\zeta)^{\frac{a_1}{2}-1}(1-\zeta)^{\frac{a_2}{2}-1}\mathsf{d}\zeta \right) \tag{6.134}$$

$$\log D_{\pm}^{(1)}(\lambda) \asymp \lambda \int_{1}^{\infty} \left((\zeta)^{\frac{a_1}{2}-1}(\zeta-1)^{\frac{a_2}{2}-1} - \frac{1}{\zeta} \right) \mathsf{d}\zeta \tag{6.135}$$

The integrals can be evaluated explicitly. The final answer reads

$$\log D_{\pm}^{(i)} = -\lambda(\gamma_E + \psi(a_j/2)) + O(\lambda^{-1})\,, \qquad j = 3 - i \tag{6.136}$$

6.5.3 Quasiclassical asymptotics for massive case

The leading large-θ asymptotics for massive problem can be found from conformal ones. Indeed, in this limit terms proportional to $\mathsf{e}^{-\theta}$ in (6.5) vanish, and the problem reduces to conformal one provided substitution

$$\lambda = \rho \mathsf{e}^{\theta} \tag{6.137}$$

is made. Taking into account slightly different normalization (6.98) of massive Wronskians we obtain

$$\begin{aligned} \log\left(\mathsf{Z}(\mathsf{e}^{\theta})\, \mathsf{W}_{\sigma}^{(i)}(\mathsf{e}^{\theta})\right)\Big|_{\mathrm{Re}(\theta)\to\pm\infty} &= 2\rho\, C_{-1}^{(i)} \cosh(\theta) - \sigma|k_{3-i}|\,\theta - \tfrac{1}{2}\log\sin(\pi|k_{3-i}|) \\ &\pm \tfrac{1}{4}\sigma \log\left(\mathfrak{S}_{3-i}\right) + C_1\, \mathsf{e}^{\mp\theta} + O\left(\mathsf{e}^{\mp 2\theta/(1+\delta)}\right) \end{aligned} \tag{6.138}$$

where constants $\mathfrak{S}_1$, $\mathfrak{S}_2$ are given by

$$\mathfrak{S}_1^{\frac{1}{2}} = \left(\frac{\rho}{a_1}\right)^{-4\frac{p_1}{a_1}} \frac{\Gamma(1+\frac{2p_1}{a_1})}{\Gamma(1-\frac{2p_1}{a_1})} \frac{1}{2p_1} \mathsf{e}^{\eta_{reg}^{(1)}} \left|\frac{z_{23}}{z_{21}z_{13}}\right|^{-2p_1} \tag{6.139}$$

$$\mathfrak{S}_2^{\frac{1}{2}} = \left(\frac{\rho}{a_2}\right)^{-4\frac{p_2}{a_2}} \frac{\Gamma(1+\frac{2p_2}{a_2})}{\Gamma(1-\frac{2p_2}{a_2})} \frac{1}{2p_2} \mathsf{e}^{\eta_{reg}^{(2)}} \left|\frac{z_{31}}{z_{32}z_{21}}\right|^{-2p_2} \tag{6.140}$$

and leading coefficients read

$$C_{-1}^{(i)} = -\gamma_E - \psi(a_j/2)\,, \tag{6.141}$$

Note that all dependence of functions $\mathsf{W}^{(i)}$ on points z_1, z_2, z_3 is embedded into definition of constants $\mathfrak{S}_i$.

6.5.4 Asymptotic analysis of Bethe ansatz equations

Functional relations (6.99a-6.99g) can be used to establish asymptotics of functions $\mathtt{W}^{(i)}$ in the rest of complex plane and the pattern of zeroes. Indeed, using the fact that $\mathtt{W}^{(3)}_{\sigma'\sigma}(\lambda) = \mathtt{W}^{(3)}_{\sigma'\sigma}(-\lambda)$ we conclude that for $\frac{\pi}{2} <\mid \mathrm{Arg}\ \lambda \mid\leq \pi$ the asymptotic of $\mathtt{W}^{(3)}_{\sigma'\sigma}(\lambda)$ is given by

$$\begin{aligned}\mathtt{W}^{(3)}_{\sigma'\sigma}(\lambda) = \Gamma\left(1+\frac{2\sigma p_1}{a_1}\right)\Gamma\left(1+\frac{2\sigma' p_2}{a_2}\right)\frac{\sqrt{a_1a_2}}{2\pi} \\ \times\left(-\frac{\lambda}{a_1}\right)^{-\frac{2\sigma p_1}{a_1}}\left(-\frac{\lambda}{a_2}\right)^{-\frac{2\sigma' p_2}{a_2}}\exp\left(-\frac{\pi\lambda}{\cos(\frac{1}{2}\pi\delta)}\right)\end{aligned} \tag{6.142}$$

Now substituting λ such that $\pi(1-\frac{1}{2}\delta) <\mid \mathrm{Arg}\ -\lambda \mid\leq \pi$ into (6.99f) and using asymptotics (6.129) and (6.133) for $\mathtt{W}^{(3)}$ and $\mathtt{W}^{(1)}$ respectively we find that for $\pi(1-\frac{1}{2}\delta) < \mathrm{Arg}\,(-\lambda) \leq \pi$ the behaviour of $\mathtt{W}^{(2)}$ is described by the following asymptotics:

$$\mathtt{W}^{(2)}_{\sigma'}(\lambda) = 2\mathrm{i}\Gamma\left(1+\frac{2\sigma' p_1}{a_1}\right)\sqrt{-\frac{a_1\lambda}{\pi}}\left(-\frac{\lambda}{a_1}\right)^{-\frac{2\sigma' p_1}{a_1}}\cos\left(\frac{2\pi p_2}{a_2}\right)\exp\left(\tilde{C}_2\lambda-2\pi\mathrm{i}\lambda\right) \tag{6.143}$$

Similarly for $-\pi(1-\frac{1}{2}\delta) > \mathrm{Arg}\,(-\lambda) \geq$ we get

$$\mathtt{W}^{(2)}_{\sigma'}(\lambda) = -2\mathrm{i}\Gamma\left(1+\frac{2\sigma' p_1}{a_1}\right)\sqrt{-\frac{a_1\lambda}{\pi}}\left(-\frac{\lambda}{a_1}\right)^{-\frac{2\sigma' p_1}{a_1}}\cos\left(\frac{2\pi p_2}{a_2}\right)\exp\left(\tilde{C}_2\lambda+2\pi\mathrm{i}\lambda\right) \tag{6.144}$$

Constant $\tilde{C}_2$ above is given by

$$\tilde{C}_2 = -\gamma_E - \psi(\tfrac{1}{2}a_1) - 2\pi\tan(\tfrac{1}{2}\pi\delta) \tag{6.145}$$

Thus we have found asymptotical expressions for all functions $\mathtt{W}^{(i)}$ in entire complex plane except few rays where these functions have zeroes or poles. The asymptotical formulae for location of those zeroes and poles can also be derived from functional relations. Indeed, asymptotical expressions do not describe behaviour of $\mathtt{W}^{(1)}$ for $\mathrm{Arg}\ \lambda = \pi$. If we plug it into (6.99f) we can replace $\mathtt{W}^{(2)}$ and $\mathtt{W}^{(3)}$ by appropriate asymptotical formulae (6.129), (6.142) and (6.133). Firstly we observe that $\mathtt{W}^{(1)}$ has poles exactly at

$$\lambda = \tfrac{1}{2} - n\,, \qquad n\in\mathbb{N} \tag{6.146}$$

n	Numerical zeroes	Asymptotic formula
1	-0.32118644	-0.28671875
2	-0.78656554	-0.78671875
3	-1.28902812	-1.28671875
4	-1.78780916	-1.78671875
5	-2.28754568	-2.28671875

Table 6.1: Zeroes of $\mathtt{W}_+^{(1)}$ computed for $\delta = \frac{17}{47}$, $p_1 = \frac{1}{10}$, $p_2 = \frac{1}{20}$

n	Numerical zeroes	Asymptotic formula
1	-0.41729701-0.37695011i	-0.44259358-0.39285900i
2	-1.12935605-0.82780500 i	-1.15306097-0.84640446i
3	-1.83960684-1.28084953i	-1.86352835-1.29994992i
4	-2.54998314-1.73418285i	-2.57399573-1.75349537i
5	-3.28446342-2.20704109i	-3.28446311-2.20704083i
6	-3.99491662-2.66056721i	-3.99493049-2.66058629i
7	-4.70539788-3.11413172i	-4.70539787-3.11413175i
8	-5.41586525-3.56767720i	-5.41586525-3.56767720i
9	-6.12633265-4.02122266i	-6.12633263-4.02122266i

Table 6.2: Zeroes of $\mathtt{W}_+^{(2)}$ computed for $\delta = \frac{17}{47}$, $p_1 = \frac{1}{10}$, $p_2 = \frac{1}{20}$

Secondly, zeroes of $\mathtt{W}_+^{(1)}$ are located approximately at

$$\lambda^{(1)}_{\sigma,n} = -\frac{1}{2}\left(n - \frac{1}{2} + \frac{2\sigma p_2}{a_2}\right) + O(n^{-1}) \tag{6.147}$$

First few zeroes are given in the table below Similarly, from functional relation (6.99f) and asymptotics we find zeroes of $\mathtt{W}^{(2)}$:

$$\lambda^{(2)}_{\sigma,n} = -\mathsf{e}^{\pm\frac{\mathrm{i}\pi\delta}{2}}\cos(\tfrac{1}{2}\pi\delta)\left(n - \frac{1}{2} + \frac{\sigma p_1}{a_1} + \frac{|p_2|}{a_2} \pm \frac{\mathrm{i}\log 2}{2\pi}\right) + O(n^{-1}) \tag{6.148}$$

Functions $\mathtt{W}^{(2)}$ also have poles at the points (6.146). Finally, functional relation (6.99g) can be used to find zeroes of $\mathtt{W}^{(3)}$, which are located at $\mathrm{Arg}\,\lambda = \pm\frac{1}{2}\pi$:

$$\lambda^{(3)}_{\sigma\sigma',n} = \pm\mathrm{i}\cos(\tfrac{1}{2}\pi\delta)\left(n - \frac{1}{2} + \frac{\sigma p_1}{a_1} + \frac{\sigma' p_2}{a_2}\right) + O(n^{-1}) \tag{6.149}$$

Right hand side of (6.99g) has other zeroes, which are zeroes of $\mathtt{T}^{(3)}$:

$$\lambda^{(*)}_{n} = -\frac{\exp(\pm\frac{1}{2}\mathrm{i}\pi\delta)}{2}\left(n - s \pm \frac{\mathrm{i}\log 2}{2\pi}\right) + O(n^{-1}) \tag{6.150}$$

where s is some real number. For the purpose of derivation of NLIE (see chapter 7)

n	Numerical zeroes	Asymptotic formula
1	0.58940086i	0.58444921i
2	1.42941557i	1.42734148i
3	2.27132015i	2.27023375i
4	3.11381245i	3.11312602i
5	3.95650092i	3.95601829i
6	4.79927351i	4.79891057i
7	5.642208861i	5.64180284i
8	6.48492773i	6.48469511i
9	7.32778156i	7.32758738i
10	8.17064498i	8.17047965i

Table 6.3: Zeroes of $\mathtt{W}_{++}^{(3)}$ computed for $\delta = \frac{17}{47}$, $p_1 = \frac{1}{10}$, $p_2 = \frac{1}{20}$

k	Numerical zeroes
1	-0.200415± 0.221398i
2	-0.583872±0.474295i
3	-1.01067 ± 0.743051i
4	-1.43127 ±1.01137i
5	-1.8275 ± 1.28017i

Table 6.4: Zeroes of $\mathtt{T}^{(3)}$ computed for $\delta = \frac{17}{47}$, $p_1 = \frac{1}{10}$, $p_2 = \frac{1}{20}$

it is useful to rewrite Bethe ansatz equations (6.77),(6.79) in the form

$$f_{12}(\lambda) = \mathsf{e}^{2\pi \mathrm{i}(-p_1+p_2+\lambda)} \frac{\mathtt{W}_{++}^{(3)}(\mathrm{i}\lambda q)}{\mathtt{W}_{++}^{(3)}(\mathrm{i}\lambda/q)} = -1 \tag{6.151}$$

$$f_3(\lambda) = \mathsf{e}^{-2\pi \mathrm{i}(p_1+p_2)+\pi \mathrm{i}(q+q^{-1})\lambda} \frac{\mathtt{W}_{+}^{(1)}(\lambda/q)\mathtt{W}_{+}^{(2)}(\lambda q)}{\mathtt{W}_{+}^{(1)}(\lambda q)\mathtt{W}_{+}^{(2)}(\lambda/q)} = -1 \tag{6.152}$$

To solve Bethe ansatz equations we need to know the asymptotical behaviour of functions f_{12}, f_3, in particular whether their absolute values decrease with absolute value of λ. From asymptotics (6.129),(6.142), (6.133) we obtain

$$f_{12}(\lambda) = \mathsf{e}^{\frac{4\pi \mathrm{i} p_2}{a_2}} \tag{6.153}$$

Above ray $\mathrm{Arg}\,(\lambda) = \frac{1}{2}\pi\delta$ we have

$$f_{12}(\lambda) = \exp\left(2\mathrm{i}\pi\lambda\left(1 - \mathrm{i}\tan(\tfrac{1}{2}\pi\delta)\right) - \frac{2\mathrm{i}\pi p_1}{a_1} + \frac{2\mathrm{i}\pi p_2}{a_2}\right) \tag{6.154}$$

This expression decreases rapidly with $|\lambda|$. Below ray $\mathrm{Arg}\,\lambda = -\frac{1}{2}\pi\delta$ we obtain

$$f_{12}(\lambda) = \exp\left(2\mathrm{i}\pi\lambda\left(1 + \mathrm{i}\tan(\tfrac{1}{2}\pi\delta)\right) - \frac{2\mathrm{i}\pi p_1}{a_1} + \frac{2\mathrm{i}\pi p_2}{a_2}\right) \tag{6.155}$$

In left half plane inside wedge ($\pi - \frac{1}{2}\pi\delta <|$ Arg λ $|< \pi$) asymptotic reads

$$f_{12}(\lambda) = \mathsf{e}^{4\mathrm{i}\pi\lambda + \frac{4\mathrm{i}\pi p_2}{a_2}} \tag{6.156}$$

This grows below real axis and decreases above it. For $|$ Arg λ $|< \frac{1}{2}\pi(1-\delta)$

$$f_3(\lambda) = \exp\left(\frac{2\pi\mathrm{i}\lambda}{\cos(\frac{1}{2}\pi\delta)} - \frac{2\pi\mathrm{i}p_1}{a_1} - \frac{2\pi\mathrm{i}p_2}{a_2}\right) \tag{6.157}$$

In wedge near imaginary axis in upper half plane $\frac{1}{2}\pi(1-\delta) <$ Arg $\lambda < \frac{1}{2}\pi(1+\delta)$

$$f_3(\lambda) = \exp\left(2\pi\mathrm{i}\lambda\mathsf{e}^{\frac{\mathrm{i}\pi\delta}{2}}\left(1 - \mathrm{i}\tan(\tfrac{1}{2}\pi\delta)\right) - \frac{2\pi\mathrm{i}p_1}{a_1} - \frac{2\pi\mathrm{i}p_2}{a_2}\right) \tag{6.158}$$

In sector $\pi(1-\delta) <$ Arg $\lambda < \pi(1-\frac{1}{2}\delta)$

$$f_3(\lambda) = \cos\left(\frac{2\pi p_2}{a_2}\right)\exp\left(4\pi\mathrm{i}\lambda\mathsf{e}^{\frac{\mathrm{i}\pi\delta}{2}} - \frac{2\pi\mathrm{i}p_2}{a_2}\right) \tag{6.159}$$

In sector $\pi(1-\frac{1}{2}\delta) <$ Arg $\lambda < \pi$

$$f_3(\lambda) = 2\mathsf{e}^{\frac{-6\pi\mathrm{i}p_2}{a_2}}\cos\left(\frac{2\pi p_2}{a_2}\right) \tag{6.160}$$

So $f_3(\lambda)$ decreases in sector $0 <$ Arg $\lambda < \pi(1-\frac{1}{2}\delta)$ and is almost constant in sector $\pi(1-\frac{1}{2}\delta) <$ Arg $\lambda < \pi$.

6.5.5 Pattern of zeroes from numerical solution

Asymptotical expressions derived in previous subsections are unsuitable for finding first few zeroes of functions $\mathtt{W}^{(i)}$ (i.e. those closest to $\lambda = 0$), especially at small values of δ. In order to find the position of these zeroes we use Mathematica to solve the ODE numerically for different values of λ. Then we compute numerically Wronskians of these solutions. We make contour plots of functions $\mathtt{W}^{(i)}$ to find starting point for root finding routine and apply that to find more precise numerical values of roots. The ODE used to define $\mathtt{W}^{(2)}$ reads

$$-y''(z) + V(z)y(z) = 0 \tag{6.161}$$

where potential is given by

$$V(z) = \frac{\lambda^2}{(1-z)^2 z^{2-a_1}} + \frac{p_1^2 - \frac{1}{4}}{z^2(1-z)} - \frac{1}{4z(1-z)^2} + \frac{\frac{1}{4} - p_2^2}{z(1-z)} \tag{6.162}$$

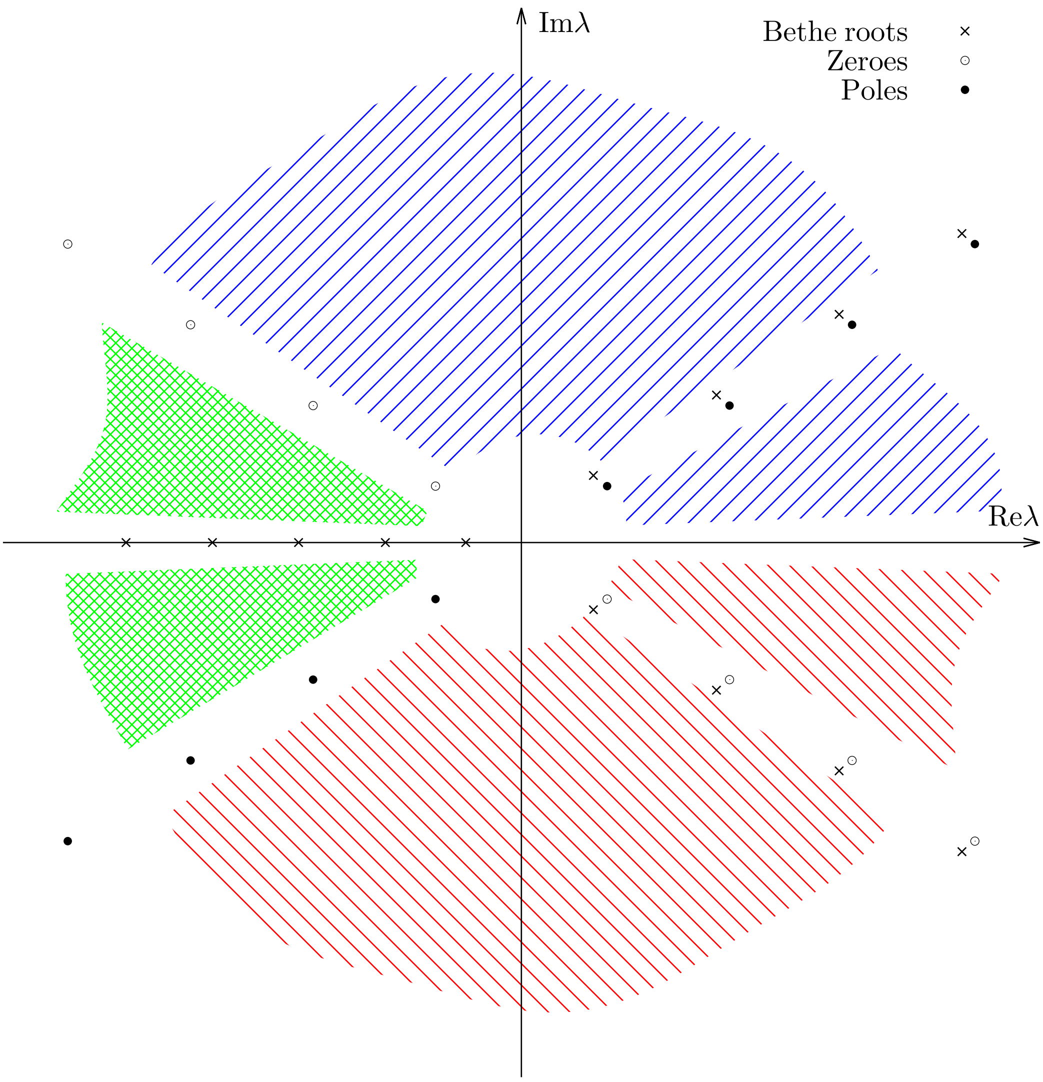


Figure 6.2: Analytical structure of $f_{12}(\lambda)$. The poles of the function are denoted by dots, the zeroes are represented by circles and the roots of equation $f_{12}(\lambda) = -1$ are depicted by crosses. The fill pattern schematically represents the asymptotical behaviour of $|f_{12}(\lambda)|$: the areas filled with (blue) right-inclined lines represent the sectors where $|f_{12}(\lambda)|$ decays asymptotically as function of $|\lambda|$ with fixed $\operatorname{Arg}\lambda$, (red) left-inclined lines correspond to growth of the function and (green) rhombic grid corresponds to existence of finite non-zero limit.

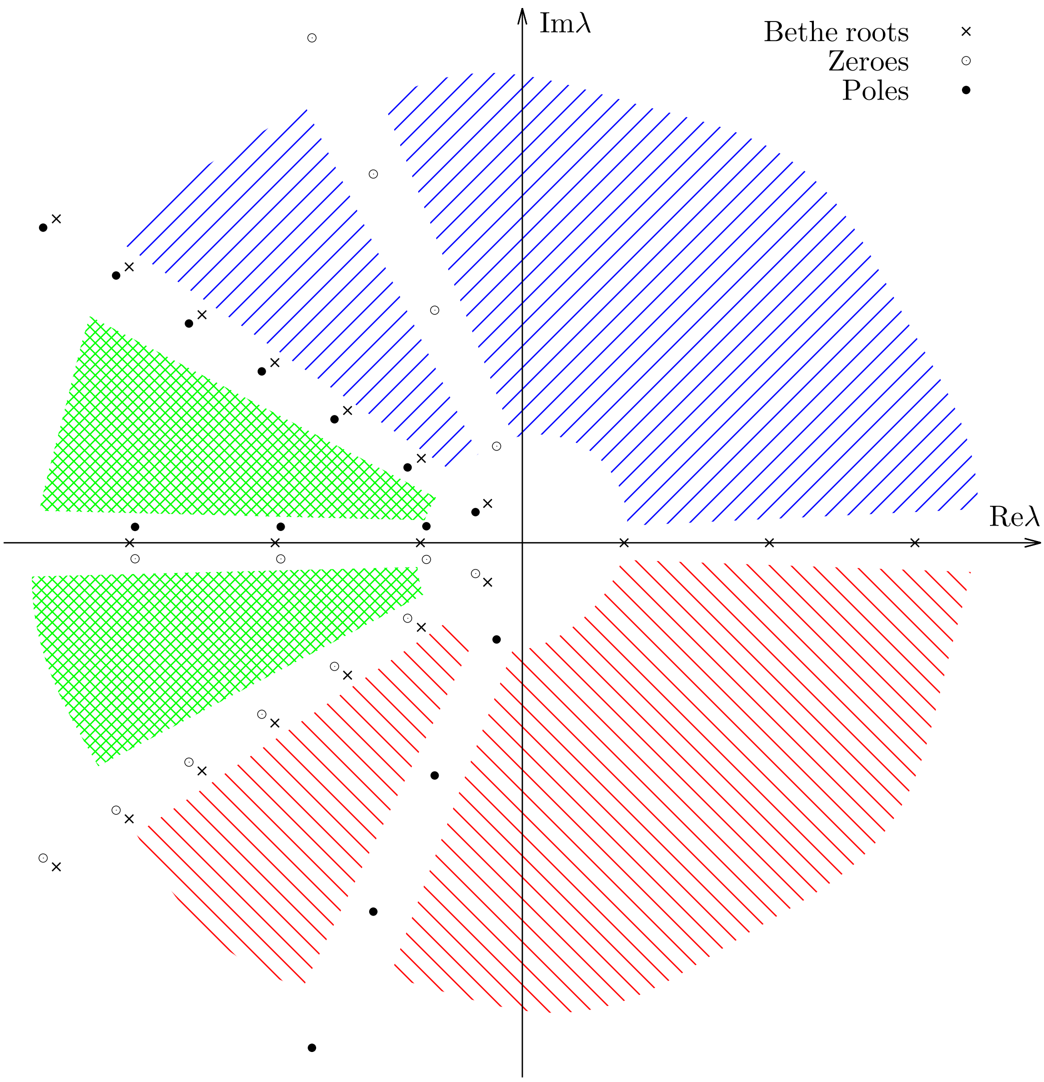
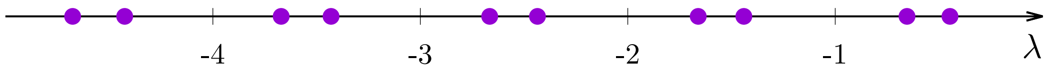


Figure 6.3: Analytical structure of f_3 The poles of the function are denoted by dots, the zeroes are represented by circles and the roots of equation $f_3(\lambda) = -1$ are depicted by crosses. Note that some of these roots are zeroes of $\mathtt{T}^{(3)}(\lambda)$, so they are "fake" Bethe roots. The fill pattern describes the asymptotical behaviour of $|f_3(\lambda)|$: the areas filled with (blue) right-inclined lines represent the sectors where $|f_3(\lambda)|$ decays asymptotically as function of $|\lambda|$ with fixed $\operatorname{Arg}\lambda$, (red) left-inclined lines correspond to growth of the function and (green) rhombic grid corresponds to existence of finite non-zero limit.

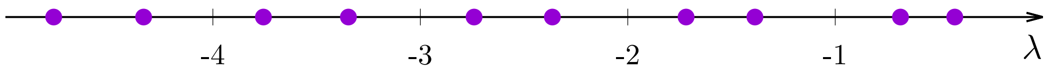
The algorithm of computing numerical values of functions $\mathtt{W}^{(1)}$ and $\mathtt{W}^{(2)}$ is as follows:

- Consider ODE (6.161) on a circle of some finite radius r, where it assumes the form
$$\frac{1}{r^2}\mathsf{e}^{-2\mathrm{i}t}(y''(t)-\mathrm{i}y'(t))+V(1-r\mathsf{e}^{\mathrm{i}t})y(t)=0 \tag{6.163}$$
- Pick two sets of initial conditions $y_0(0)=1$, $y_0'(0)=0$ and $y_1(0)=0$, $y_1'(0)=1$
- Solve numerically for $y_0(t)$ and $y_1(t)$ on interval $[0:2\pi]$
- Compute monodromy matrix
$$M=\begin{pmatrix} y_0(2\pi) & y_1(2\pi) \\ y_0'(2\pi) & y_1'(2\pi) \end{pmatrix} \tag{6.164}$$
- Diagonalise it to find eigenfunctions $y_\pm^{(3)}$ on a circle
- Consider ODE (6.161) on an interval $[\varepsilon_0;1-\varepsilon_1]$ (in practice $\varepsilon_0=10^{-40}$, $\varepsilon_1=0.01$).
- Solve it numerically using $y_\pm^{(3)}(0)$ to set initial conditions at $z=1-r$
- Take solutions for equation with $\lambda=0$ as an approximation for $y_\pm^{(2)}(z)$ in vicinity of $z=0$
$$y_\pm^{(1)}=z^{\frac{1}{2}\pm p_1}\sqrt{1-z}\,{}_2F_1(\tfrac{1}{2}\pm p_1-p_2,\tfrac{1}{2}\pm p_1+p_2,1\pm 2p_1;z) \tag{6.165}$$
- Compute Wronskian at the point $z=\varepsilon_0$
- We've obtained $\mathtt{W}_\pm^{(1)}(\lambda)$

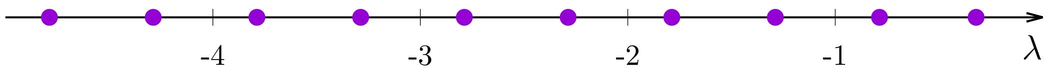
In practice we observe that for $|\lambda|\gg 5$ the error of that calculation grows insanely. In that region we are better served with asymptotic expressions obtained above. For the purpose of finding zeroes of functions $\mathtt{W}^{(1)}$, $\mathtt{W}^{(2)}$ it is more convenient to solve equation (6.151) which involves functions $\mathtt{W}^{(3)}$, because function f_{12} does not have poles in vicinity of Bethe roots, which is not the case for $\mathtt{W}^{(1)}$. For $\delta=0$ zeroes of $\mathtt{W}^{(1)}$ and $\mathtt{W}^{(2)}$ are arranged in pairs on real axis in accordance with formula (6.105). As we increase parameter δ the functions $\mathtt{W}^{(1)}$ and $\mathtt{W}^{(2)}$ start to behave differently: zeroes of $\mathtt{W}^{(1)}$, grouped in pairs for small δ (see Fig.6.4a), begin to move apart (Fig. 6.4b) till in leading order they are evenly spaced (6.147) (see Fig.6.4c). The process starts from infinity, i.e. for sufficiently small δ the zeroes will be arranged in pairs for $|\lambda|\sim 1$ and even-spaced for $|\lambda|\gg 1$. The more we increase δ, the closer the area with new pattern approaches zero. On the other hand, zeroes of $\mathtt{W}^{(2)}$, also grouped



(a) Zeroes of $\mathtt{W}_+^{(1)}$ at $\delta = 0.003$, $p_1 = 0.1$, $p_2 = 0.05$



(b) Zeroes of $\mathtt{W}_+^{(1)}$ at $\delta = 0.03$, $p_1 = 0.1$, $p_2 = 0.05$



(c) Zeroes of $\mathtt{W}_+^{(1)}$ at $\delta = \frac{17}{47}$, $p_1 = 0.1$, $p_2 = 0.05$

Figure 6.4: Positions of zeroes of $\mathtt{W}_+^{(1)}$ for different values of δ

in pairs, move closer to each other till they coincide and then they move to complex plane, forming 2-strings along lines $\operatorname{Arg}\lambda = \pm\pi(1-\frac{\delta}{2})$ in accordance with (6.148), as depicted on Fig.6.6. Again, for small enough δ the zeroes near $\lambda = 0$ still lie on real axis (typical pattern is shown on Fig. 6.5), while for $|\lambda| \gg 1$ form 2-strings For larger δ all zeroes of $\mathtt{W}^{(1)}$ would be almost evenly spaced, and all zeroes of $\mathtt{W}^{(2)}$ would lie on 2-strings. In order to compute $\mathtt{W}_{\sigma'\sigma}^{(3)}$ numerically from ODE it is convenient to consider different form of this equation

$$-\partial_y^2\chi(y) + \left(\frac{p_1^2}{4(1+\mathsf{e}^y)} + \frac{p_2^2\mathsf{e}^y}{4(1+\mathsf{e}^y)} - \frac{\mathsf{e}^{ya_1}\lambda^2}{4(1+\mathsf{e}^y)^2} + \frac{\mathsf{e}^y}{4(1+\mathsf{e}^y)^2}\right)\chi(y) = 0 \tag{6.166}$$

The algorithm is as follows

- Consider equation (6.166) on interval $[-L; L]$ (for practical purposes $L = 60$)

- Define solutions y_1, y_2 by initial conditions

$$y_1(-L) = y_1^{(0)}(-L) \qquad y_1^{'}(-L) = (y_1^{(0)})'(-L) \tag{6.167}$$

$$y_2(L) = y_2^{(0)}(L) \qquad y_2^{'}(L) = (y_2^{(0)})'(L) \tag{6.168}$$

  where $y_1^{(0)}$ and $y_2^{(0)}$ are solutions to equation with $\lambda = 0$ (see expressions below).

- Solve the ODE numerically using NDSolve[] function on $[-L; L]$

- Compute Wronskian $\mathtt{W}_{\sigma'\sigma}^{(3)}(\mathrm{i}\lambda) = [y_1, y_2]$ at $y = +L$

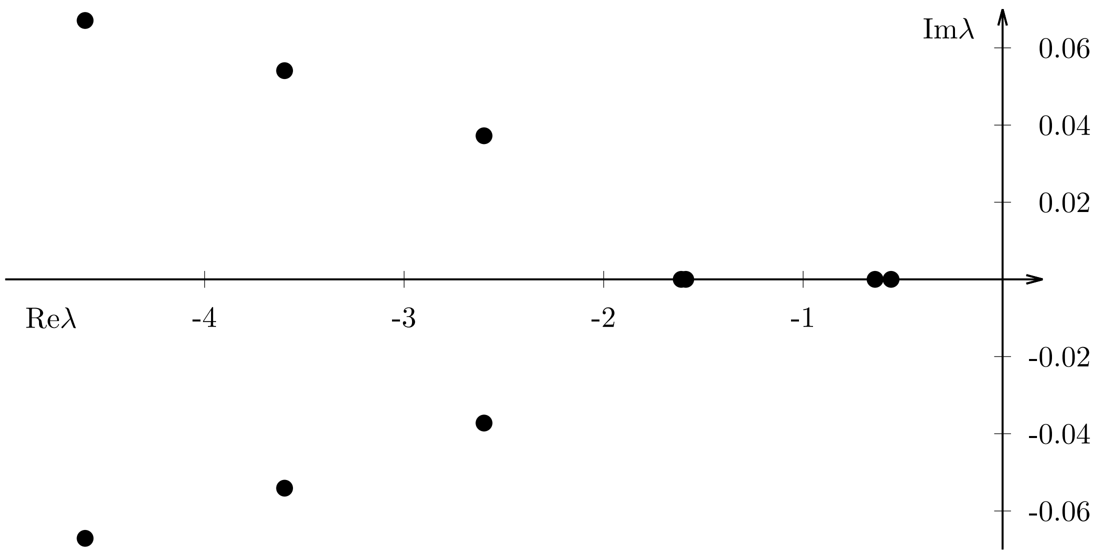


Figure 6.5: Zeroes of $\mathtt{W}_+^{(2)}$ at $\delta = 0.003$, $p_1 = 0.1$, $p_2 = 0.05$

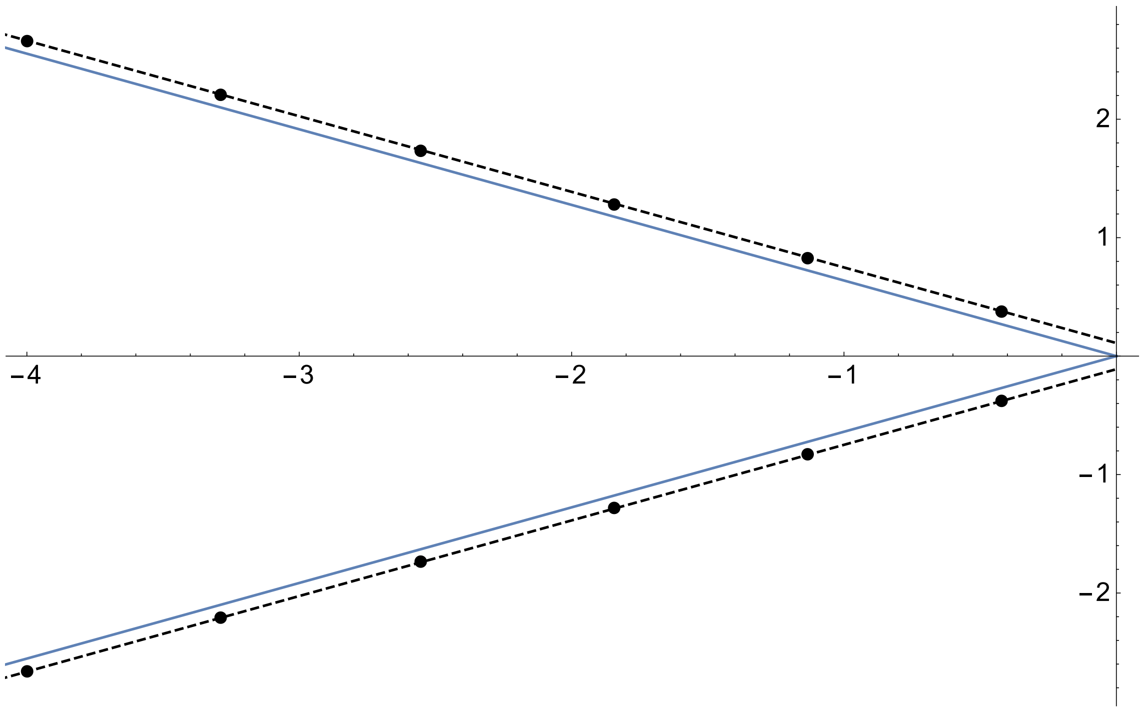


Figure 6.6: Zeroes of $\mathtt{W}^{(2)}$ at $\delta = \frac{17}{47}$, $p_1 = 0.1$, $p_2 = 0.05$

$$
\begin{aligned}
y_\sigma^{(1)}(y) &= \mathsf{e}^{\sigma p_1}(1+\mathsf{e}^y)^{-\sigma p_1 - p_2}{}_2F_1\left(\frac{1}{2}+\sigma p_1 + p_2, \frac{1}{2}+\sigma p_1 + p_2, 1+2\sigma p_1; \frac{\mathsf{e}^y}{1+\mathsf{e}^y}\right) \\
y_\sigma^{(2)}(y) &= \mathsf{e}^{p_1}(1+\mathsf{e}^y)^{-p_1 - \sigma' p_2}{}_2F_1\left(\frac{1}{2}+p_1 + \sigma' p_2, \frac{1}{2}+p_1 + \sigma' p_2, 1+2\sigma' p_2; \frac{1}{1+\mathsf{e}^y}\right)
\end{aligned}
$$

Zeroes of $\mathtt{W}^{(3)}$ are located on imaginary axis and obey the expression (6.149). As $\delta \to 0$ it assumes its limiting form (6.109), so there is nothing particularly interesting in intermediate regime.

Chapter 7

NLIE for vacuum state

7.1 Derivation of Non-Linear Integral Equations

In this section we are going to derive a system of non-linear integral equations equivalent to the whole system of Bethe ansatz equations. This derivation relies on prior knowledge of approximate positions of roots of Bethe ansatz. Hence we start with considering conformal case, for which we have asymptotical expressions and approximate values for first few roots obtained from numerical solution of ODE. We shall argue then that only minor modifications of this method are required to obtain NLIE for the massive case. The NLIE in question are akin to those derived by Destri and DeVega in [88, 89] and Batchelor et al. in [90]. In the case considered here, however, the derivation is significantly more involved and merits detailed presentation.

Firstly let us set $\sigma, \sigma', \sigma''$ in functional relations (6.70-6.73) to $+$. Throughout this section we shall omit these indices, assuming they all are set to $+$ unless otherwise specified. Consider $\lambda^{(1)}$ such that $\mathtt{W}_{+}^{(1)}(\lambda^{(1)}) = 0$. Then equation (6.70) can be rewritten as

$$-1 = \mathsf{e}^{2\pi \mathrm{i}(-p_1+p_2+\lambda^{(1)})} \frac{\mathtt{W}^{(3)}(\mathrm{i}\lambda^{(1)}q)}{\mathtt{W}^{(3)}(\mathrm{i}\lambda^{(1)}/q)} \tag{7.1}$$

It is convenient to introduce notation for the function in right hand side:

$$f_{12}(\lambda) = \mathsf{e}^{2\pi \mathrm{i}(-p_1+p_2+\lambda)} \frac{\mathtt{W}^{(3)}(\mathrm{i}\lambda q)}{\mathtt{W}^{(3)}(\mathrm{i}\lambda/q)} \tag{7.2}$$

If we take $\lambda^{(2)}$ to be zero of $\mathtt{W}_{+}^{(2)}$ instead, we'll get the condition

$$f_{12}(-\lambda^{(2)}) = -1 \tag{7.3}$$

Finally, plugging $\lambda^{(3)}$ such that $\mathtt{W}_{++}^{(3)}(\mathrm{i}\lambda^{(3)}) = 0$ into equation (6.73) we get condition

$$f_3(\lambda^{(3)}) = -1 \tag{7.4}$$

where function $f_3(\lambda)$ is defined as

$$f_3(\lambda) = \mathsf{e}^{-2\pi\mathrm{i}(p_1+p_2)+\pi\mathrm{i}(q+q^{-1})\lambda}\frac{\mathtt{W}^{(1)}(\lambda/q)\mathtt{W}^{(2)}(\lambda q)}{\mathtt{W}^{(1)}(\lambda q)\mathtt{W}^{(2)}(\lambda/q)} \tag{7.5}$$

From construction (6.58)-(6.60) it is clear that functions $\mathtt{W}^{(1)}(\lambda)$, $\mathtt{W}^{(2)}(\lambda)$, $\mathtt{W}^{(3)}(\mathrm{i}\lambda)$ are real on the real axis. They are also analytical functions of λ in right half plane. Then the following properties of functions f_{12}, f_3 can be established:

$$f_{12}(\lambda^*)f_{12}^*(\lambda) = 1\,, \qquad f_3(\lambda^*)f_3^*(\lambda) = 1 \tag{7.6}$$

Equations for zeroes of functions $\mathtt{W}_+^{(1)}$, $\mathtt{W}_+^{(2)}$, $\mathtt{W}_{++}^{(3)}$ then form a closed system of equations which can be summarized as follows:

- If $\mathtt{W}_{++}^{(3)}(\mathrm{i}\lambda) = 0$ then $f_3(\lambda) + 1 = f_3(-\lambda) + 1 = 0$
- If $\mathtt{W}_+^{(1)}(\lambda) = 0$ then $f_{12}(\lambda) + 1 = 0$
- If $\mathtt{W}_+^{(2)}(\lambda) = 0$ then $f_{12}(-\lambda) + 1 = 0$

We shall show that this system, combined with analytical properties of functions $\mathtt{W}^{(i)}$ and their asymptotics determines these functions completely.

Since functions $\mathtt{W}^{(1)}(\lambda)/\Gamma(1+2\lambda)$, $\mathtt{W}^{(2)}(\lambda)/\Gamma(1+2\lambda)$ and $\mathtt{W}^{(3)}(\mathrm{i}\lambda)$ are entire functions of λ, we can make use by the following theorem due to Weierstrass [91] (quoted from [92])

Theorem 1. *Let g be an entire function that can have zero of order m at the point $z = 0$, $\{z_k\}$ a set of it's zeroes different from $z = 0$. Then there exist an entire function h and a set of integer numbers p_i such that*

$$g(z) = z^m e^{h(z)} \prod_{i=1}^{\infty} E_{p_i}(z/z_i) \tag{7.7}$$

where elementary factors E_k are given by

$$E_k = (1-z)\exp\left(\sum_{j=1}^{k} \frac{z^j}{j}\right) \tag{7.8}$$

In our case the asymptotics of the logarithms of entire functions in question are linear, and consequently the asymptotic density of their zeroes is constant. Functions f_{12} and f_3 introduced above are ratios of entire functions and we can use the Weierstrass theorem to represent their logarithms as a sums over zeroes of functions $\mathtt{W}^{(i)}$:

$$\log f_{12}(\lambda) = 2\pi\mathrm{i}(-p_1+p_2+\lambda) + \sum_{\lambda_k \in \mathrm{Ze}_3^+} \left(\mathcal{F}(\lambda/\lambda_k) + \mathcal{F}(-\lambda/\lambda_k)\right) \tag{7.9}$$

Here Ze_3^+ is a discrete set of zeroes of $\mathtt{W}^{(3)}(\mathrm{i}\lambda)$ such that $\mathrm{Re}\,\lambda > 0$. In writing (7.9) we have used the fact that $\mathtt{W}^{(3)}(\lambda)$ is a function of λ^2 to account for all the zeroes. Function $\mathcal{F}$ appearing in r.h.s. of (7.9) is given by

$$\mathcal{F}(\lambda) = \log\left(\frac{1-\lambda q}{1-\lambda/q}\right) \tag{7.10}$$

Similarly we can expand logarithm of $f_3(\lambda)$. Note that, even though functions $\mathtt{W}^{(1)}$, $\mathtt{W}^{(2)}$ are not entire functions themselves, we can still represent f_3 as a ratio of entire functions by multiplying and dividing by same product of gamma functions, so poles of $\mathtt{W}^{(i)}$ do not contribute to f_3:

$$\begin{aligned}\log f_3(\lambda) = -2\pi\mathrm{i}\left(p_1+p_2-\cos\left(\tfrac{1}{2}\pi\delta\right)\lambda\right)+c\lambda+&\sum_{\lambda_k\in\mathrm{Ze}_2}\left(\mathcal{F}(\lambda/\lambda_k)+2\mathrm{i}\sin\left(\tfrac{1}{2}\pi\delta\right)\frac{\lambda}{\lambda_k}\right)\\ -&\sum_{\lambda_k\in\mathrm{Ze}_1}\left(\mathcal{F}(\lambda/\lambda_k)+2\mathrm{i}\sin\left(\tfrac{1}{2}\pi\delta\right)\frac{\lambda}{\lambda_k}\right)\end{aligned} \tag{7.11}$$

Constant c is given by

$$c = \frac{2\pi f_{12}'(0)}{1+f_{12}(0)} \tag{7.12}$$

Sets Ze_1 and Ze_2 are sets of all zeroes of $\mathtt{W}^{(1)}(\lambda)$ and $\mathtt{W}^{(2)}(\lambda)$ respectively.

We can rewrite sums as contour integrals using the following observation: function $\partial_\lambda \log(1+f_{12}(\lambda))$ has a pole of residue one whenever equation $f_{12}(\lambda) = -1$ is satisfied. Then sum of values of some function $K(\lambda)$ over zeroes of $\mathtt{W}^{(1)}_{++}$ can be written as

$$\sum_{\lambda_k\in\mathrm{Ze}_1} K(\lambda_k) = \frac{1}{2\pi\mathrm{i}}\oint_{\mathcal{C}_1} K(\lambda)\partial_\lambda\log(1+f_{12}(\lambda))\mathsf{d}\lambda \tag{7.13}$$

provided all zeroes of $\mathtt{W}^{(1)}_{++}$ lie inside the contour $\mathcal{C}_1$ and integrand doesn't have any other poles inside the contour. Such poles may occur when

- There are solutions to equation $f_{12}(\lambda) = -1$ that do not correspond to zeroes of $W^{(1)}$
- Points $f_{12}(\lambda) = \infty$ are poles of residue -1
- Kernel (function K) has its own singularities inside the contour

Since we know that all zeroes of $\mathtt{W}^{(3)}_{++}(\mathrm{i}\lambda)$ lie on real axis, and the function depends only on λ^2, we can write the following integral representation for $f_{12}(\lambda)$

$$\log f_{12}(\lambda) = X_{12}(\lambda) - \frac{1}{2\pi\mathrm{i}}\oint_{\mathcal{C}_3}\partial_\mu\left(\mathcal{F}(\lambda\mu^{-1})+\mathcal{F}(-\lambda\mu^{-1})\right)\log(1+f_3(\mu))\mathsf{d}\mu \tag{7.14}$$

where contour $\mathcal{C}_3$ encircles all the zeroes of $\mathtt{W}_{++}^{(3)}$ anticlockwise in the right half-plane. Deriving this formula we have integrated by parts. No boundary terms arise since integrand decreases sufficiently fast at large $|\mu|$. Consider singularities of the integrand. Due to functional relation (6.73), roots of equation (7.5) that do not correspond to zeroes of $\mathtt{W}_{++}^{(3)}$ must be zeroes of $\mathtt{T}^{(3)}$. According to (6.150) they lie in left half-plane. Function f_3 has poles where $\mathtt{W}_{+}^{(1)}(\lambda q)$ or $\mathtt{W}_{+}^{(2)}(\lambda/q)$ have zeroes. It is evident that these poles are restricted to domain $|\mathrm{Arg}\ \mu| > \frac{\pi}{2}$. Finally, kernel is singular when

$$\mu = \sigma\lambda\,\mathrm{e}^{\frac{\mathrm{i}\sigma'\pi\delta}{2}} \tag{7.15}$$

Let us assume that λ and each point μ from contour $\mathcal{C}_3$ satisfy one of the following sets of conditions

$$|\mathrm{Arg}\,(-\lambda)| < \tfrac{1}{2}\pi\delta\,, \qquad |\mathrm{Arg}\,\mu| < \tfrac{1}{2}\pi\delta - |\mathrm{Arg}\,(-\lambda)| \tag{7.16}$$

$$\tfrac{1}{2}\pi\delta < |\mathrm{Arg}\,\lambda| < \tfrac{1}{2}\pi\,, \qquad |\mathrm{Arg}\,\mu| < |\mathrm{Arg}\,\lambda| - \tfrac{1}{2}\pi\delta \tag{7.17}$$

Then no unwanted singularity falls inside the contour and we have

$$X_{12} = 2\pi\mathrm{i}(-p_1 + p_2 + \lambda) \tag{7.18}$$

Similarly we can write down integral representation for $f_3(\lambda)$:

$$\log f_3(\lambda) = X_3(\lambda) + U_1(\lambda) + U_2(\lambda) \tag{7.19}$$

with

$$U_1(\lambda) = -\frac{1}{2\pi\mathrm{i}} \oint\limits_{\mathcal{C}_1} \partial_\mu \mathcal{F}_{\mathrm{r}}(\lambda\mu^{-1}) \log(1 + f_{12}(\mu))\mathsf{d}\mu \tag{7.20}$$

$$U_2(\lambda) = -\frac{1}{2\pi\mathrm{i}} \oint\limits_{\mathcal{C}_2} \partial_\mu \mathcal{F}_{\mathrm{r}}(-\lambda\mu^{-1}) \log(1 + f_{12}(\mu))\mathsf{d}\mu \tag{7.21}$$

where contours $\mathcal{C}_1$ and $\mathcal{C}_2$ encircle zeroes of $\mathtt{W}_{+}^{(1)}(\lambda)$ and $\mathtt{W}_{+}^{(2)}(-\lambda)$ respectively, and $\mathcal{F}_{\mathrm{r}}$ stands for

$$\mathcal{F}_{\mathrm{r}}(\lambda) = \mathcal{F}(\lambda) + 2\mathrm{i}\lambda \sin\left(\tfrac{1}{2}\pi\delta\right) \tag{7.22}$$

In this case, provided there are no zeroes of $\mathtt{W}_{+}^{(2)}(-\lambda)$ inside $\mathcal{C}_1$ and no zeroes of $\mathtt{W}_{+}^{(1)}(\lambda)$ inside $\mathcal{C}_2$, there are no other solutions of (7.1), (7.3) contributing to the integrals. This condition is trivially satisfied. However, $f_{12}(\lambda)$ has poles where $\mathtt{W}_{++}^{(3)}(\mathrm{i}\lambda q)$ has zeroes, and half of such points fall unavoidably inside $\mathcal{C}_2$. These points, in accordance with previous considerations, have residues -1. We can compute their contribution as an integral over $\mathcal{C}_3$ following similar steps as when deriving (7.14). Then we

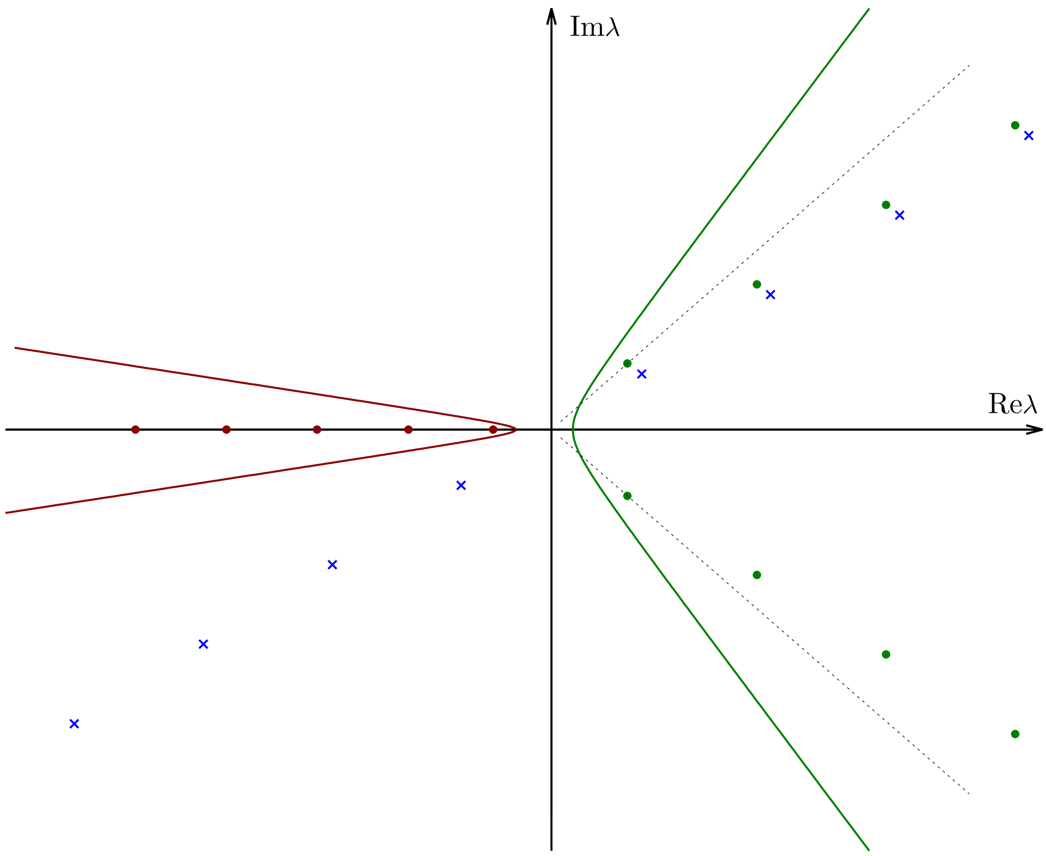


Figure 7.1: Contours $\mathcal{C}_1$ and $\mathcal{C}_2$ (conformal case). Solid lines represent the contours. Dashed lines represent the minimal cone that contains all zeroes of $\mathtt{W}^{(2)}(-\lambda)$. Red dots represent zeroes of $\mathtt{W}^{(1)}$. Green dots represent zeroes of $\mathtt{W}^{(2)}$. Blue crosses denote poles of function $f_{12}(\lambda)$

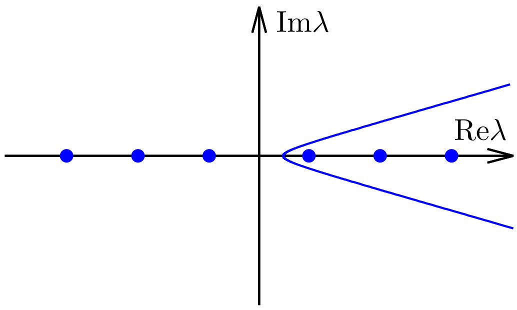


Figure 7.2: Contour $\mathcal{C}_3$ (conformal case). Blue dots represent zeroes of $\mathtt{W}^{(3)}(\mathrm{i}\lambda)$.

obtain

$$X_3(\lambda) = X_{30}(\lambda) + U_3(\lambda) \tag{7.23}$$

$$U_3(\lambda) = \frac{1}{2\pi\mathrm{i}} \oint\limits_{\mathcal{C}_3} \partial_\mu \mathcal{F}_{\mathrm{r}}(-\lambda\mu^{-1}\mathrm{e}^{-\frac{\mathrm{i}\pi\delta}{2}})\partial_\mu \log(1+f_3(\mu))\mathrm{d}\mu \tag{7.24}$$

Finally we consider singularities of kernel $\mathcal{F}_{\mathrm{r}}$. They give rise to conditions

$$|\mathrm{Arg}\,\mu| > \tfrac{1}{2}\pi\delta + |\mathrm{Arg}\,\lambda|\,, \quad \mu \in \mathcal{C}_1 \tag{7.25}$$

$$|\mathrm{Arg}\,\mu| < \pi - \tfrac{1}{2}\pi\delta - |\mathrm{Arg}\,\lambda|\,, \quad \mu \in \mathcal{C}_2 \tag{7.26}$$

$$|\mathrm{Arg}\,\mu| < \pi(1-\delta) - \mathrm{Arg}\,\lambda|\,, \quad \mu \in \mathcal{C}_3 \tag{7.27}$$

ensuring that there are no extra singularities inside relevant contours. It is easy to check that provided $|\mathrm{Arg}\,\lambda| < \frac{1}{2}\pi(1-\delta)$ it is possible to draw contours $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ respecting (7.25) and encircling respective zeroes. As a result the following representation is obtained

$$\log f_3(\lambda) = -2\pi\mathrm{i}\left(p_1 + p_2 - \lambda\cos\left(\tfrac{1}{2}\pi\delta\right)\right) + U_1(\lambda) + U_2(\lambda) + U_3(\lambda) \tag{7.28}$$

Since both analytical and numerical treatment of irregularly shaped contours is complicated, we shall set them to be straight lines. However, point $\mu = 0$ is a pole of function $\mathcal{F}_{\mathrm{r}}(\lambda\mu^{-1})$, and we can't extend our contours to this point. So we specify each of our contours to consist of three segments:

$$\mathcal{C}_i = \Gamma_i^{(+)} \cup \Gamma_i^{(-)} \cup \Gamma_i^{(0)} \tag{7.29}$$

with

$$\Gamma_i^{(\pm)} : |\lambda| \geq \varepsilon \;\; \mathrm{Arg}(\lambda) = \pm\omega_i \tag{7.30}$$

and

$$\Gamma_i^{(0)} : |\lambda| = \varepsilon \;\; \mathrm{Arg}(\lambda) \in [-\omega_i; \omega_i] \;\; i = 2,3 \tag{7.31}$$

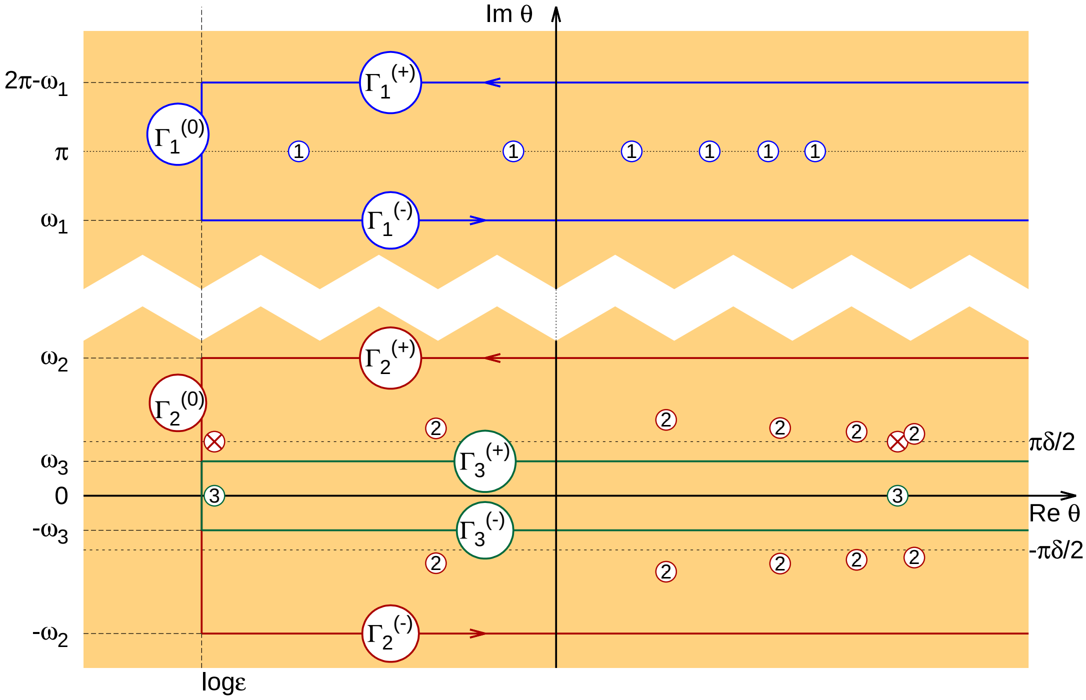


Figure 7.3: Contours $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ in θ plane. Encircled numbers denote zeroes of $\mathtt{W}^{(1)}(\mathsf{e}^{\theta})$, $\mathtt{W}^{(2)}(-\mathsf{e}^{\theta})$, $\mathtt{W}^{(3)}(\mathrm{i}\mathsf{e}^{\theta})$. Crosses stand for "fake roots"(zeroes of $\mathtt{T}^{(3)}$).

$$\Gamma_1^{(0)} : |\lambda| = \varepsilon \ \ \mathrm{Arg}(\lambda) \in [-\pi; -\omega_1] \cup [\omega_1; \pi] \tag{7.32}$$

It is convenient to introduce functions $\epsilon^{(i)}$

$$\epsilon^{(1)}(\theta) = -\mathrm{i}\log f_{12}(\mathsf{e}^{\theta-\mathrm{i}\omega_1}) \tag{7.33}$$

$$\epsilon^{(2)}(\theta) = -\mathrm{i}\log f_{12}(\mathsf{e}^{\theta-\mathrm{i}\omega_2}) \tag{7.34}$$

$$\epsilon^{(3)}(\theta) = -\mathrm{i}\log f_{3}(\mathsf{e}^{\theta-\mathrm{i}\omega_3}) \tag{7.35}$$

From the properties (7.6) of functions f_{12}, f_3 it immediately follows that

$$f_{12}(\mathsf{e}^{\theta+\mathrm{i}\omega_1}) = \mathsf{e}^{\mathrm{i}\epsilon_1^*(\theta)} \tag{7.36}$$

$$f_{12}(\mathsf{e}^{\theta+\mathrm{i}\omega_2}) = \mathsf{e}^{\mathrm{i}\epsilon_2^*(\theta)} \tag{7.37}$$

$$f_{3}(\mathsf{e}^{\theta+\mathrm{i}\omega_3}) = \mathsf{e}^{\mathrm{i}\epsilon_3^*(\theta)} \tag{7.38}$$

Then U_1 can be computed as

$$\begin{aligned}2\pi\mathrm{i}U_1(\lambda) = 2\pi\mathrm{i}Y_1 + \mathrm{i}\int_{\varepsilon}^{\infty}\partial_\theta\mathcal{F}_{\mathrm{r}}(\lambda\mathsf{e}^{-\theta+\mathrm{i}\omega_1})\epsilon_1(\theta)\mathsf{d}\theta + \int_{\varepsilon}^{\infty}\partial_\theta\mathcal{F}_{\mathrm{r}}(\lambda\mathsf{e}^{-\theta+\mathrm{i}\omega_1})b_1(\theta)\mathsf{d}\theta \\ - \int_{\varepsilon}^{\infty}\partial_\theta\mathcal{F}_{\mathrm{r}}(\lambda\mathsf{e}^{-\theta-\mathrm{i}\omega_1})b_1^*(\theta)\mathsf{d}\theta\end{aligned} \tag{7.39}$$

where notation

$$b_1(\theta) = \log(1 + \mathsf{e}^{-\mathrm{i}\epsilon^{(1)}(\theta)}) \tag{7.40}$$

was introduced and Y_1 stands for contribution of circular part of the contour $\mathcal{C}_1$:

$$Y_1 = -\frac{1}{2\pi\mathrm{i}} \int\limits_{\omega_1}^{2\pi-\omega_1} \partial_s \mathcal{F}_{\mathrm{r}}(\lambda\varepsilon^{-1}\mathsf{e}^{-\theta-\mathrm{i}s}) \log(1 + f_{12}(\varepsilon\mathsf{e}^{\theta+\mathrm{i}s})\mathsf{d}s \tag{7.41}$$

Parameter ε stands for radius of the circular segment of the contour, and U_1 does not depend on it provided it satisfies

$$0 < \varepsilon < \min|\lambda_k^{(1)}| \tag{7.42}$$

Then we can take limit $\varepsilon \to +0$. The idea of the following derivation is to split r.h.s. of (7.39) (and similar representations for U_2, U_3) into singular and regular parts in ε. Then singular parts must cancel each other, and regular parts can be continued analytically to $\varepsilon = 0$. For brevity sake we provide detailed workings for U_1 only, since for U_2 and U_3 we must make exactly the same steps.

Since function b_1 decays as $\exp(-4\mathrm{i}\pi\mathsf{e}^{\theta-\mathrm{i}\omega_1})$ for $\operatorname{Re}\theta \to +\infty$, we can get rid of regularizing term by splitting integral

$$\int\limits_{\varepsilon}^{\infty} \partial_\theta \mathcal{F}_{\mathrm{r}}(\lambda\mathsf{e}^{-\theta+\mathrm{i}\omega_1}) b_1(\theta)\mathsf{d}\theta = \int\limits_{\varepsilon}^{\infty} \partial_\theta \mathcal{F}(\lambda\mathsf{e}^{-\theta+\mathrm{i}\omega_1}) b_1(\theta)\mathsf{d}\theta - 2\mathrm{i}\lambda\sin(\tfrac{1}{2}\pi\delta) \int\limits_{\varepsilon}^{\infty} \mathsf{e}^{-\theta+\mathrm{i}\omega_1} b_1(\theta)\mathsf{d}\theta \tag{7.43}$$

The first term in r.h.s. of (7.43) has finite limit when ε tends to zero, while the second one is linear in λ with some ε-dependent coefficient. Integral involving b_1^* can be decomposed in similar way. Finally, first term in r.h.s. of (7.39) can be decomposed as

$$\begin{aligned} \int\limits_{\varepsilon}^{\infty} \partial_\theta \mathcal{F}_{\mathrm{r}}(\lambda\mathsf{e}^{-\theta+\mathrm{i}\omega_1}) \epsilon_1(\theta)\mathsf{d}\theta &= \int\limits_{\varepsilon}^{\infty} \partial_\theta \mathcal{F}(\lambda\mathsf{e}^{-\theta+\mathrm{i}\omega_1}) \left(\epsilon_1(\theta) - z_1(\theta)\right) \mathsf{d}\theta \\ - 2\mathrm{i}\lambda\sin\frac{\pi\delta}{2} \int\limits_{\varepsilon}^{\infty} \mathsf{e}^{-\theta+\mathrm{i}\omega_1} \left(\epsilon_1(\theta) - z_1(\theta)\right) \mathsf{d}\theta &+ \int\limits_{\varepsilon}^{\infty} \partial_\theta \mathcal{F}_{\mathrm{r}}(\lambda\mathsf{e}^{-\theta+\mathrm{i}\omega_1}) z_1(\theta)\mathsf{d}\theta \end{aligned} \tag{7.44}$$

where leading (i.e. non-decreasing at large θ) part of $\epsilon^{(i)}$ was denoted as

$$z_1(\theta) = 2\mathsf{e}^{\theta-\mathrm{i}\omega_1} - \mathsf{c}_1\,, \qquad z_2 = q\mathsf{e}^{\theta-\mathrm{i}\omega_2} - \mathsf{c}_2\,, \qquad z_3(\theta) = \frac{\mathsf{e}^{\theta-\mathrm{i}\omega_3}}{\cos(\frac{1}{2}\pi\delta)} - \mathsf{c}_3 \tag{7.45}$$

$$\mathsf{c}_1 = -\frac{2p_2}{a_2}\,, \qquad \mathsf{c}_2 = \frac{p_1}{a_1} - \frac{p_2}{a_2}\,, \qquad \mathsf{c}_3 = \frac{p_1}{a_1} + \frac{p_2}{a_2} \tag{7.46}$$

In the r.h.s. of last equation first term is finite at $\varepsilon = 0$, second is linear in λ and third one can be computed explicitely. For arbitrary real parameters β, ω, complex parameter α and $\sigma = \pm 1$ we have

$$\int_{\varepsilon}^{\infty} \mathsf{e}^{-\mathrm{i}\omega}\alpha t \partial_t \left(\mathcal{F}\left(\frac{\sigma\lambda}{t\mathsf{e}^{-\mathrm{i}\omega+\mathrm{i}\beta}}\right) + \frac{2\mathrm{i}\lambda\sigma\sin(\frac{1}{2}\pi\delta)}{t\mathsf{e}^{-\mathrm{i}\omega+\mathrm{i}\beta}} \right) \mathsf{d}t = 2\mathrm{i}\sigma\alpha\mathsf{e}^{-\mathrm{i}\beta}\lambda\sin(\tfrac{1}{2}\pi\delta)\log\varepsilon$$
$$+ \sigma\alpha\mathsf{e}^{-\mathrm{i}\beta}\lambda\left(-\mathsf{e}^{\frac{\mathrm{i}\pi\delta}{2}}\log(\varepsilon - \sigma\lambda\mathsf{e}^{\frac{\mathrm{i}\pi\delta}{2}+\mathrm{i}\omega-\mathrm{i}\beta}) + \mathsf{e}^{-\frac{\mathrm{i}\pi\delta}{2}}\log(\varepsilon - \sigma\lambda\mathsf{e}^{-\frac{\mathrm{i}\pi\delta}{2}+\mathrm{i}\omega-\mathrm{i}\beta})\right) \tag{7.47}$$

$$\int_{\varepsilon}^{\infty} C\partial_t \left(\mathcal{F}\left(\frac{\sigma\lambda}{t\mathsf{e}^{-\mathrm{i}\omega+\mathrm{i}\beta}}\right) + \frac{2\mathrm{i}\lambda\sigma\sin(\frac{1}{2}\pi\delta)}{t\mathsf{e}^{-\mathrm{i}\omega+\mathrm{i}\beta}} \right) \mathsf{d}t = C\mathcal{F}_{\mathrm{r}}(\lambda\varepsilon^{-1}\sigma\mathsf{e}^{\mathrm{i}\omega-\mathrm{i}\beta}) \tag{7.48}$$

In small ε limit only terms proportional to λ and $\lambda\log\lambda$ will survive:

$$\int_{\varepsilon}^{\infty} \mathsf{e}^{-\mathrm{i}\omega}\alpha t\partial_t\mathcal{F}_{\mathrm{r}}\left(\frac{\sigma\lambda}{t\mathsf{e}^{-\mathrm{i}\omega+\mathrm{i}\beta}}\right)\mathsf{d}t = -\sigma\alpha\mathsf{e}^{-\mathrm{i}\beta}2\mathrm{i}\sin(\tfrac{1}{2}\pi\delta)\lambda\log\lambda + \kappa\lambda + O(\varepsilon) \tag{7.49}$$

Coefficient κ in above expression depends on ε and diverges in small ε limit. From (7.41), (7.10) and (7.2) we compute expansion of Y_1 at small ε:

$$Y_1 = -2\lambda\sin\frac{\pi\delta}{2}\frac{\sin\omega_1}{\pi}\log(1+f_{12}(0))\varepsilon^{-1} - 2\lambda\sin\frac{\pi\delta}{2}\frac{\pi-\omega_1}{\pi}\frac{f'_{12}(0)}{1+f_{12}(0)} + O(\varepsilon^2) \tag{7.50}$$

We see that terms surviving in the limit $\varepsilon \to 0$ are linear in λ. We repeat this procedure for U_2 and U_3. At the end of the day all integrals that are not known as functions of λ are finite at $\varepsilon = 0$. Since $\epsilon^{(3)}$ as a whole is independent of ε, all divergent terms in r.h.s. should sum up into finite quantities, which consist of linear in λ and constant term. But integrals with $\mathcal{F}$ tend to zero at large λ and consequently do not contain non-decreasing terms in large λ expansion, so unknown coefficients are fixed by known asymptotics of $\epsilon^{(i)}$. In the same way we transform integral representations for $\epsilon^{(1)}$, $\epsilon^{(2)}$. Resulting system of NLIE can be written as

$$\epsilon_i(\theta) - \sum_j \int_{-\infty}^{\infty} \mathsf{d}\theta'\, R_{ij}^{(+)}(\theta-\theta')\,\epsilon_j(\theta') = 2\pi\, P_i(\theta) - 2\pi\,\mathsf{C}_i \tag{7.51}$$
$$+ \frac{1}{\mathrm{i}}\sum_j \int_{-\infty}^{\infty} R_{ij}^{(+)}(\theta-\theta')\,\log\left(1+\mathsf{e}^{-\mathrm{i}\varepsilon_j(\theta')}\right)\mathsf{d}\theta'$$
$$- \frac{1}{\mathrm{i}}\sum_j \int_{-\infty}^{\infty} R_{ij}^{(-)}(\theta-\theta')\,\log\left(1+\mathsf{e}^{-\mathrm{i}\varepsilon_j^*(\theta')}\right)\mathsf{d}\theta',$$

with kernel R defined by its Fourier transform:

$$R_{ij}^{(\pm)}(\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\nu\theta}\hat{R}_{ij}^{(\pm)}(\nu)d\nu \tag{7.52}$$

$$\hat{R}_{ij}^{(\pm)}(\nu) = d_i(\nu)\hat{R}_{ij}(\nu)d_j(\mp\nu) \tag{7.53}$$

$$d_1(\nu) = \mathsf{e}^{(\pi-\omega_1)\nu} \qquad d_2(\nu) = \mathsf{e}^{-\omega_2\nu} \qquad d_3(\nu) = \mathsf{e}^{-\omega_3\nu} \tag{7.54}$$

$$\begin{aligned}
\hat{R}_{11}(\nu) &= \hat{R}_{12}(\nu) = 0\,, & \hat{R}_{13}(\nu) &= -\frac{\sinh(\frac{\pi\nu(1-\delta)}{2})}{\sinh(\frac{\pi\nu}{2})} \\
\hat{R}_{21}(\nu) &= \hat{R}_{22}(\nu) = 0\,, & \hat{R}_{23}(\nu) &= \frac{\sinh(\frac{\pi\nu\delta}{2})}{\sinh(\frac{\pi\nu}{2})}\mathsf{e}^{-\frac{\pi\nu}{2}} \\
\hat{R}_{31}(\nu) &= \hat{R}_{32}(\nu) = \frac{\sinh(\frac{\pi\nu\delta}{2})}{\sinh(\pi\nu)} & \hat{R}_{33}(\nu) &= \frac{\sinh(\frac{\pi\nu\delta}{2})}{\sinh(\pi\nu)}\,\mathsf{e}^{\frac{\pi\nu\delta}{2}}\,.
\end{aligned}$$

Source terms in (7.51) are given by

$$\mathsf{C}_1 = \mathsf{C}_2 = p_1 - p_2\,, \qquad \mathsf{C}_3 = p_1 + p_2 \tag{7.55}$$

and

$$P_1(\theta) = P(\theta - \mathrm{i}\omega_1, -\delta)\,, \qquad P_2(\theta) = P(\theta - \mathrm{i}\omega_2, -\delta)\,, \qquad P_3(\theta) = P(\theta - \mathrm{i}\omega_3, 0)\,. \tag{7.56}$$

In conformal case considered here

$$P(\theta,\alpha) = \frac{\cos(\frac{1}{2}\pi\alpha)}{\cos(\frac{1}{2}\pi\delta)}\mathsf{e}^{\theta} \tag{7.57}$$

For lattice-regularized case (see remark at the end of subsection) expressions for $P(\theta,\delta)$ are given by (5.55). Equations (7.51) can be further simplified if we invert integral operator acting on left hand side. We get

$$\begin{aligned}
\epsilon_i(\theta) &= 2\pi z_i(\theta) + \frac{1}{\mathrm{i}}\sum_{j=1}^{3}\int_{-\infty}^{\infty}\mathsf{d}\theta'\, G_{ij}^{(+)}(\theta-\theta')\ \log\left(1+\mathsf{e}^{-\mathrm{i}\epsilon_j(\theta')}\right) \\
&\quad -\frac{1}{\mathrm{i}}\sum_{j=1}^{3}\int_{-\infty}^{\infty}\mathsf{d}\theta'\, G_{ij}^{(-)}(\theta-\theta')\ \log\left(1+\mathsf{e}^{-\mathrm{i}\epsilon_j^*(\theta')}\right)\,,
\end{aligned} \tag{7.58}$$

where $G^{(\pm)}$ is related to G similarly to (7.53,7.54) and Fourier transform of kernel G can be computed as

$$\hat{G}(\nu) = (1-\hat{R}(\nu))^{-1}\hat{R}(\nu) \tag{7.59}$$

the final expression in components reads ($i = 1, 2$)

$$\hat{G}_{1i}(\nu) = -\frac{\sinh(\frac{\pi\nu\delta}{2})}{2\cosh(\frac{\pi\nu}{2})\sinh(\frac{\pi\nu a_2}{2})}\,, \qquad \hat{G}_{13}(\nu) = -\frac{\sinh(\frac{\pi\nu}{2})}{\sinh(\frac{\pi\nu a_2}{2})}$$

$$\hat{G}_{2i}(\nu) = \frac{\sinh^2(\frac{\pi\nu\delta}{2})\,\mathsf{e}^{-\frac{\pi\nu}{2}}}{2\,\cosh(\frac{\pi\nu}{2})\sinh(\frac{\pi\nu a_1}{2})\sinh(\frac{\pi\nu a_2}{2})}\,, \quad \hat{G}_{23}(\nu) = \frac{\sinh(\frac{\pi\nu}{2})\,\sinh(\frac{\pi\nu\delta}{2})\,\mathsf{e}^{-\frac{\pi\nu}{2}}}{\sinh(\frac{\pi\nu a_1}{2})\sinh(\frac{\pi\nu a_2}{2})}$$

$$\hat{G}_{3i}(\nu) = \frac{\sinh(\frac{\pi\nu}{2})\,\sinh(\frac{\pi\nu\delta}{2})}{2\,\cosh(\frac{\pi\nu}{2})\sinh(\frac{\pi\nu a_1}{2})\sinh(\frac{\pi\nu a_2}{2})}\,, \quad \hat{G}_{33}(\nu) = \frac{\sinh^2(\frac{\pi\nu\delta}{2})}{\sinh(\frac{\pi\nu a_1}{2})\sinh(\frac{\pi\nu a_2}{2})}\,.$$

From functional relations (6.70)-(6.73) we can obtain different closed systems of BAE governing positions of zeroes of $\mathtt{W}^{(1)}_{-}$, $\mathtt{W}^{(2)}_{-}$, $\mathtt{W}^{(3)}_{--}$ and $\mathtt{W}^{(3)}_{+-}$ by taking different pairs of indices σ, σ'. These systems, however, will only differ from (7.1), (7.3), (7.4) by signs in front of p_1, p_2 in the source term. The leading term in asymptotical formulae for positions of zeroes is unaffected by such change. Therefore, in order to obtain NLIE describing these zeroes it is enough to alter aforementioned signs in final answer (7.58).

The derivation above can be generalised to massive case. In order to do so we replace representation (7.7) with (5.81), (5.87). Note that due to different normalization of solutions to massive problem functions $\mathsf{W}^{(i)}(\mathsf{e}^{\theta})$ do not have any poles, though they have an essential singularity at $\lambda = \mathsf{e}^{\theta} = 0$. We can decompose these representations as

$$\mathsf{W}^{(i)}(\theta) = \mathsf{W}^{(i)}_{pos}(\mathsf{e}^{\theta})\mathsf{W}^{(i)}_{neg}(\mathsf{e}^{-\theta}) \tag{7.60}$$

where $\mathsf{W}^{(i)}_{pos}$ and $\mathsf{W}^{(i)}_{neg}$ are entire functions admitting representation (7.7). The division between zeroes $\theta^{(i)}_{+}$ and $\theta^{(i)}_{-}$ is arbitrary as long as there exists θ_{max} such that

$$\forall k \;\; \theta^{(i)}_{k,+} > -\theta_{max} \qquad \forall k \;\; \theta^{(i)}_{k,-} < \theta_{max} \tag{7.61}$$

Given that the system enjoys mass gap it is convenient to set that $\mathsf{e}^{\theta^{(i)}_k}$ is a zero of $\mathsf{Q}^{(i)}_{pos}$ if and only if Re $(\theta^{(i)}_k) > 0$. The contour entering integral representations of sums over $\theta^{(i)}_k$ then will be a union

$$\mathcal{C}_i = \mathcal{C}^{(+)}_i \cup \mathcal{C}^{(-)}_i \tag{7.62}$$

where contour $\mathcal{C}^{(+)}_i$ ($\mathcal{C}^{(-)}_i$) winds counterclockwise around all zeroes $\theta^{(i)}_{k,+}$ ($\theta^{(i)}_{k,-}$) and leaves the other half of zeroes outside. When we replace them with straight lines, we shall have integrals over $[0 : \infty)$ and $(-\infty, 0]$ respectively which can be, after subtraction of linear terms in e^{θ} and $\mathsf{e}^{-\theta}$ can be merged into expressions exactly similar to integrals in (7.51). The segment $\Gamma^{(0)}$ will be now

$$\operatorname{Re}\theta = 0 \qquad \operatorname{Im}\theta \in [-\omega_i, \omega_i] \tag{7.63}$$

and integral over it can be demonstrated to be a linear combination of the form

$$a\,\mathsf{e}^{\theta} + b + c\,\mathsf{e}^{-\theta} \tag{7.64}$$

with some θ-independent coefficients a, b, c (precise values of those constants are irrelevant as total contributions of this kind are known from asymptotics). Also note that those a, b, c should not be mistaken for any other quantities denoted by these letters. The functions $b^{(i)}$ will now decay at both infinities exponentially in $\mathsf{e}^{-\mathsf{e}^{|\theta|}}$ Thus, except for the source term

$$z_i(\theta) \mapsto \mathbf{z}_i(\theta) \tag{7.65}$$

$$\mathbf{z}_1(\theta) = \frac{r}{\pi}\cos(\tfrac{1}{2}\pi\delta)\sinh(\theta - \mathrm{i}\omega_1) - \mathsf{c}_1 \tag{7.66}$$

$$\mathbf{z}_2(\theta) = \frac{r}{\pi}\cos(\tfrac{1}{2}\pi\delta)\sinh(\theta - \mathrm{i}\omega_2 + \tfrac{1}{2}\mathrm{i}\pi\delta) - \mathsf{c}_2 \tag{7.67}$$

$$\mathbf{z}_3(\theta) = \frac{r}{2\pi}\sinh(\theta - \mathrm{i}\omega_3) - \mathsf{c}_3 \tag{7.68}$$

the NLIE (7.58) will remain the same with exactly the same kernel G. To avoid unnecessary repetition we will not provide detailed workings for the massive case here, but instead we shall provide a detailed derivation of integral representations of functions $\mathsf{Q}^{(i)}$ themselves on the example of massive case. This derivation will capture all the subtleties related to merging contours and getting read of regularising terms.

Finally, NLIE (7.58) can be modified to solve lattice-regularised system of BAE (5.31), (5.56). One notices that it differs from (7.1), (7.3) only in the form of source term. The crucial difference, which makes derivation of NLIEs even simpler in this case, is the fact that the number of zeroes is finite, and therefore no Weierstrass multiplier is required in representation (7.7) (elementary factors E_k are rational). Consequently, function $\mathcal{F}_{\mathrm{r}}$ can be replaced by $\mathcal{F}$ in (7.11) without making the sum divergent. As there is no singularity at $\lambda = 0$ the contours can be continued to this point from the very beginning. Two subtleties have to be observed. Firstly, the source term now has branch cuts in complex plane

$$\theta = \pm\Theta + \frac{\mathrm{i}\pi a_1}{2} \tag{7.69}$$

thus, in addition to conditions (7.16), (7.25) one has to ensure contours do not cross them. However, in practice this does not restrict further the range of suitable values of parameters ω_i. Secondly, functions $b^{(i)}$ in this case have constant non-zero limits at both infinities, and $\epsilon^{(i)}$ will also have constant limits. So we should subtract

$$\begin{aligned} \mathsf{z}_1(\theta) &= -N\mathsf{z}_\theta(\theta - \mathrm{i}\omega_1 + \mathrm{i}\pi) - \mathsf{c}_1 \\ \mathsf{z}_2(\theta) &= N\mathsf{z}_u(\theta - \mathrm{i}\omega_2 + \tfrac{1}{2}\mathrm{i}\pi\delta) - \mathsf{c}_2 \\ \mathsf{z}_3(\theta) &= N\mathsf{z}_u(\theta - \mathrm{i}\omega_3) - \mathsf{c}_3 \end{aligned} \tag{7.70}$$

At this point we can make stronger argument regarding ODE/IM correspondence. The system of lattice-regularized BAEs (5.31),(5.56) is equivalent to a system of NLIEs (7.58) with the source term (7.70). The latter in scaling limit differs from the source term for the connection coefficients of massive problem by terms decreasing as $\exp(-\Theta)$ for $\theta \ll \Theta$. Thus we expect that for any fixed θ

$$\lim_{4N\mathrm{e}^{\Theta}=r,\, \Theta\to\infty} \epsilon^{(i)}_{\mathrm{Lat}}(\theta,\Theta) \to \epsilon^{(i)}_{\mathrm{M}}(\theta) \tag{7.71}$$

Since functions $\epsilon^{(i)}$ describe distribution of zeroes we have proved the ODE/IQFT correspondence for Bukhvostov-Lipatov model.

7.2 Integral representations for sums over Bethe roots

The next step is to derive integral representation for functions W themselves. Let us introduce the following notations

$$D(\theta) = \log(1 - \mathrm{e}^{\theta}) \qquad \mathcal{D}(\theta) = D(\theta) + \mathrm{e}^{\theta} \tag{7.72}$$

Then representation (7.7, 7.60) of functions $\mathsf{W}^{(i)}(\theta)$ can be written in logarithmic form as

$$\log \mathsf{W}^{(i)} = \log \mathsf{W}^{(i)}(0) + y^{(i)} \sinh\theta + \sum \mathcal{D}(\theta - \theta^{(i)}_{k,+}) + \sum \mathcal{D}(\theta^{(i)}_{k,-} - \theta) \tag{7.73}$$

with some constant coefficients $y^{(i)}$. The sum in question is replaced by contour integrals

$$\sum_k \mathcal{D}\left(\pm(\theta - \theta^{(1)}_{k,\pm})\right) = \frac{1}{2\pi\mathrm{i}} U_{11,\pm}(\theta) \tag{7.74}$$

$$\sum_k \mathcal{D}\left(\pm(\theta - \theta^{(3)}_{k,\pm})\right) = \frac{1}{2\pi\mathrm{i}} U_{33,\pm}(\theta) \tag{7.75}$$

Where we introduced shortcut notations for individual integrals

$$U_{11,\pm} = -\oint_{\mathcal{C}_1^{(\pm)}} \partial_\tau \mathcal{D}(\pm(\theta-\tau)) \log(1+\mathsf{e}^{\mathrm{i}\epsilon^{(1)}(\tau)})\mathsf{d}\tau \tag{7.76}$$

$$U_{33,\pm} = -\oint_{\mathcal{C}_3^{(\pm)}} \partial_\tau \left(\mathcal{D}(\pm(\theta-\tau)) + \mathcal{D}(\pm(\theta-\tau+\mathrm{i}\pi))\right) \log(1+\mathsf{e}^{\mathrm{i}\epsilon^{(3)}(\tau)})r\tau \tag{7.77}$$

Contours $\mathcal{C}_i^{(+)}$ and $\mathcal{C}_i^{(-)}$ wind counterclockwise around all zeroes of $\mathsf{W}^{(i)}$ such that $\operatorname{Re}\theta_{k,+}^{(i)} > 0$ ($\operatorname{Re}\theta_{k,-}^{(i)} < 0$ respectively). In order to compute sum over zeroes $\theta^{(2)}$ one has to subtract contribution of points

$$\theta = \theta_{k,\pm}^{(3)} + \tfrac{1}{2}\mathrm{i}\pi\delta \tag{7.78}$$

which are poles of function f_{12} that lie inside the contours $\mathcal{C}_2^{(\pm)}$

$$\sum_k \mathcal{D}\left(\pm(\theta-\theta_{k,\pm}^{(2)})\right) = \frac{1}{2\pi\mathrm{i}}(U_{22,\pm}(\theta) + U_{23,\pm}(\theta)) \tag{7.79}$$

$$U_{22,\pm} = -\oint_{\mathcal{C}_2^{(\pm)}} \partial_\tau \mathcal{D}(\pm(\theta-\tau+\mathrm{i}\pi)) \log(1+\mathsf{e}^{\mathrm{i}\epsilon^{(2)}(\tau)})\mathsf{d}\tau \tag{7.80}$$

$$U_{23,\pm} = -\oint_{\mathcal{C}_3^{(\pm)}} \partial_\tau \mathcal{D}(\pm(\theta-\tau+\mathrm{i}\pi(1-\frac{\delta}{2}))) \log(1+\mathsf{e}^{\mathrm{i}\epsilon^{(3)}(\tau)})\mathsf{d}\tau \tag{7.81}$$

In order to derive convenient integral representation we have to repeat the steps we took while deriving the NLIEs in the first place: We start by specifying contours:

$$\mathcal{C}_i^{(\pm)} = \Gamma_{i,\pm}^{(+)} \cup \Gamma_{i,\pm}^{(0)} \cup \Gamma_{i,\pm}^{(-)} \tag{7.82}$$

where (for massive case presently considered)

$$\Gamma_{i,+}^{(\pm)} : \operatorname{Re}\theta \in [0:\infty),\;\; \operatorname{Im}\theta = \pm\omega_i \qquad \Gamma_{i,-}^{(\pm)} : \operatorname{Re}\theta \in (-\infty:0],\;\; \operatorname{Im}\theta = \pm\omega_i \tag{7.83}$$

$$\Gamma_i^{(0)} : \operatorname{Re}\theta = 0,\;\; \operatorname{Im}\theta \in [-\omega_i:\omega_i] \tag{7.84}$$

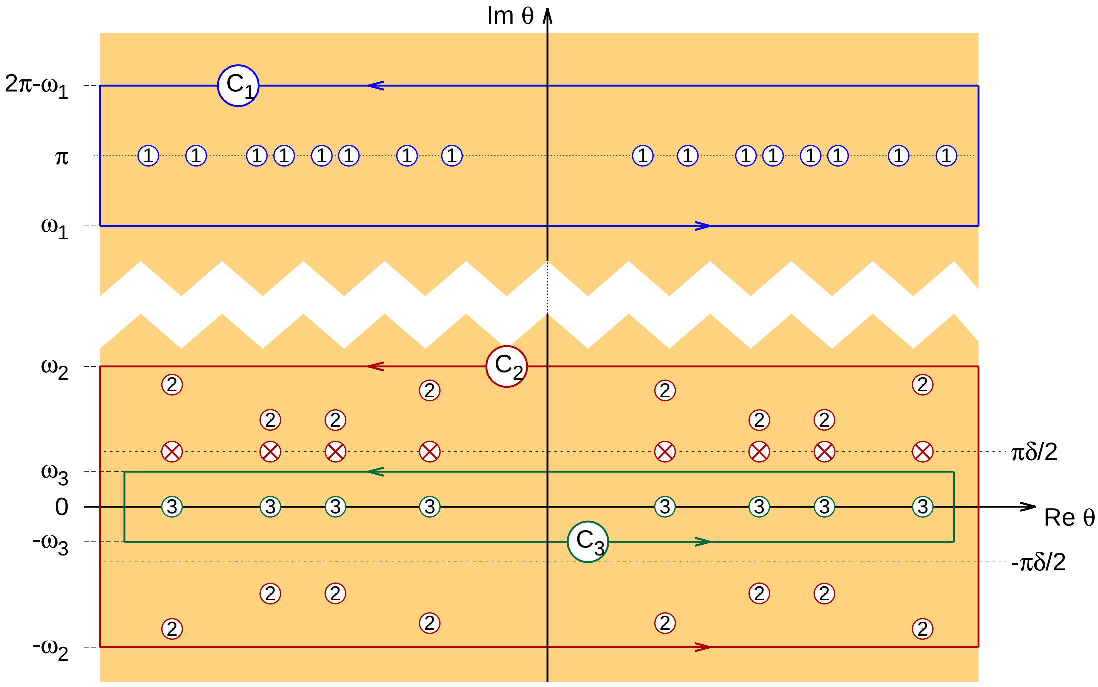


Figure 7.4: Contours $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ for lattice-regularized Bethe ansatz equations. Encircled numbers represent zeroes of functions $\mathsf{Q}^{(i)}$. Crosses represent "fake roots". For the sake of visibility for this particular plot small ($N = 4$) number of roots was taken, but for larger N pattern of zeroes is qualitatively similar and the contours should be chosen in similar fashion.

Then integrals $U_{ij}^{(\pm)}$ take form

$$
\begin{aligned}
&U_i^{(+)} + U_i^{(-)} = Y_+^{(i)}(\theta) + Y_-^{(i)}(\theta)\\
&+ \int_0^\infty \partial_\tau \mathcal{D}(\theta - \tau + \mathrm{i}\omega_i) \log(1 + \mathsf{e}^{\mathrm{i}\epsilon^{(i)}(\tau)}) \mathsf{d}\tau - \int_0^\infty \partial_\tau \mathcal{D}(\theta - \tau - \mathrm{i}\omega_i)(b^{(i)}(\tau))^* \mathsf{d}\tau \\
&+ \int_{-\infty}^0 \partial_\tau \mathcal{D}(\tau - \theta - \mathrm{i}\omega_i) \log(1 + \mathsf{e}^{\mathrm{i}\epsilon^{(i)}(\tau)}) \mathsf{d}\tau - \int_{-\infty}^0 \partial_\tau \mathcal{D}(\tau - \theta + \mathrm{i}\omega_i)(b^{(i)}(\tau))^* \mathsf{d}\tau
\end{aligned}
\tag{7.85}
$$

The difficulty here is that the contour now crosses a pole of kernel. Next step is to pull out the terms that grow exponentially at $|\mathrm{Re}\theta| \to \infty$ from logarithm

$$
\begin{aligned}
&\int_0^\infty \partial_\tau \mathcal{D}(\theta - \tau + \mathrm{i}\omega_i) \log(1 + \mathsf{e}^{\mathrm{i}\epsilon^{(i)}(\tau)}) \mathsf{d}\tau = \int_0^\infty \partial_\tau \mathcal{D}(\theta - \tau - i\omega_i) \epsilon^{(i)}(\tau) \mathsf{d}\tau \\
&+ \int_0^\infty \partial_\tau D(\theta - \tau - \mathrm{i}\omega_i) b^{(i)}(\tau) \mathsf{d}\tau - \mathsf{e}^{\theta} \int_0^\infty \mathsf{e}^{-\tau + \mathrm{i}\omega_i} b^{(i)}(\tau) \mathsf{d}\tau
\end{aligned}
\tag{7.86}
$$

For contribution of contour $\mathcal{C}_i^{(-)}$ we also split the kernel using the fact that

$$D(-\theta) = D(\theta) - \theta + \mathrm{i}\pi \tag{7.87}$$

to make it's non-trivial part coincide with the kernel for the integral over $\mathcal{C}_+^{(i)}$

$$\begin{aligned}\int_{-\infty}^{0} \partial_\tau \mathcal{D}(\tau - \theta - \mathrm{i}\omega_i) \log(1 + \mathsf{e}^{\mathrm{i}\epsilon^{(i)}(\tau)}) \mathsf{d}\tau = \int_{-\infty}^{0} \partial_\tau \mathcal{D}(\tau - \theta - \mathrm{i}\omega_i) \epsilon^{(i)}(\tau) \mathsf{d}\tau \\ + \int_{-\infty}^{0} \partial_\tau D(\theta - \tau + \mathrm{i}\omega_i) b^{(i)}(\tau) \mathsf{d}\tau - \mathsf{e}^{-\theta} \int_{-\infty}^{0} \mathsf{e}^{\tau - \mathrm{i}\omega_1} b^{(i)}(\tau) \mathsf{d}\tau + \int_{-\infty}^{0} b(\tau) \mathsf{d}\tau\end{aligned} \tag{7.88}$$

We can do it because functions $b^{(i)}$ decay exponentially in $\mathsf{e}^{|\theta|}$ for $\theta \to -\infty$, which is fast enough to suppress the growth of the new kernel. Then we rewrite the term with integrand linear in $\epsilon^{(i)}$ using NLIEs:

$$\begin{aligned}&\int_0^{\infty} \partial_\tau \mathcal{D}(\theta - \tau + \mathrm{i}\omega_i) \epsilon^{(i)}(\tau) \mathsf{d}\tau = \int_0^{\infty} \partial_\tau \mathcal{D}(\theta - \tau + \mathrm{i}\omega_i) \mathbf{z}^{(i)}(\tau) \mathsf{d}\tau \\ &+ \sum_{k=1}^{3} \int_0^{\infty} \mathsf{d}\tau \partial_\tau D(\theta - \tau + \mathrm{i}\omega_i) \int_{-\infty}^{\infty} \mathsf{d}\eta \left(G_{ik}^{(+)}(\tau - \eta) b^{(i)}(\eta) - G_{ik}^{(-)}(\tau - \eta)(b^{(i)}(\eta))^* \right) \mathsf{d}\eta \\ &- \mathsf{e}^{\theta} \sum_{k=1}^{3} \int_0^{\infty} \mathsf{d}\tau \mathsf{e}^{-\tau + i\omega_i} \int_{-\infty}^{\infty} \left(G_{ik}^{(+)}(\tau - \eta) b^{(k)}(\eta) - G_{ik}^{(-)}(\tau - \eta)(b^{(k)})^* \right) \mathsf{d}\eta\end{aligned} \tag{7.89}$$

$$\begin{aligned}&\int_{-\infty}^{0} \partial_\tau \mathcal{D}(\tau - \theta - \mathrm{i}\omega_i) \epsilon^{(i)}(\tau) \mathsf{d}\tau = \int_{-\infty}^{0} \partial_\tau \mathcal{D}(\tau - \theta - \mathrm{i}\omega_i) \mathbf{z}^{(i)}(\tau) \mathsf{d}\tau \\ &+ \sum_{k=1}^{3} \int_{-\infty}^{0} \mathsf{d}\tau \partial_\tau D(\theta - \tau + \mathrm{i}\omega_i) \int_{-\infty}^{\infty} \left(G_{ik}^{(+)}(\tau - \eta) b^{(k)}(\eta) - G_{ik}^{(-)}(\tau - \eta)(b^{(k)}(\eta))^* \right) \mathsf{d}\eta \\ &+ \int_{-\infty}^{0} \mathsf{d}\tau (1 - \mathsf{e}^{-\theta + \tau - \mathrm{i}\omega_i}) \sum_{k=1}^{3} \int_{-\infty}^{\infty} \left(G_{ik}^{(+)}(\tau - \eta) b^{(k)}(\eta) - G_{ik}^{(-)}(\tau - \eta)(b^{(k)}(\eta))^* \right) d\eta\end{aligned} \tag{7.90}$$

Integrals contributing to the leading terms in asymptotic expansion can be computed explicitly

$$\int_0^{\infty} \partial_\tau \mathcal{D}(\theta - \tau + \mathrm{i}\omega_i) \sinh(\tau - \mathrm{i}\omega_i) \mathsf{d}\tau = -\sinh(\theta) \log(1 - \mathsf{e}^{\theta}) + \frac{1}{2} + \frac{1}{4} \mathsf{e}^{\theta} \tag{7.91}$$

$$\int_{-\infty}^{0} \partial_\tau \mathcal{D}(\tau-\theta-\mathrm{i}\omega_i)\sinh(\tau-\mathrm{i}\omega_i)\mathsf{d}\tau = \sinh(\theta)\log(1-\mathsf{e}^{-\theta})+\frac{1}{2}+\frac{1}{4}\mathsf{e}^{-\theta} \tag{7.92}$$

$$\int_{0}^{\infty} \partial_\tau \mathcal{D}(\theta-\tau+\mathrm{i}\omega_i)\mathsf{d}\tau = -\log(1-\mathsf{e}^{\theta+\mathrm{i}\omega_1})-\mathsf{e}^{\theta+\mathrm{i}\omega_1} \tag{7.93}$$

$$\int_{-\infty}^{0} \partial_\tau \mathcal{D}(\tau-\theta-\mathrm{i}\omega_i)\mathsf{d}\tau = \log(1-\mathsf{e}^{-\theta-\mathrm{i}\omega_1})+\mathsf{e}^{-\theta-\mathrm{i}\omega_1} \tag{7.94}$$

Summing up contributions from all relevant contours we get

$$\log \mathsf{W}^{(i)}_{as} = \log \mathsf{W}^{(i)}(0) - \frac{r\cos\frac{\pi\delta}{2}}{\pi}\theta\sinh\theta + C^{(i)}\sinh^2\frac{\theta}{2} + -\frac{2p_{3-i}\theta}{a_{3-i}} \quad i=1,2 \tag{7.95}$$

Constants $C^{(i)}$ is non-universal: only difference is

$$C^{(2)}-C^{(1)} = r\sin\left(\frac{\pi\delta}{2}\right) \tag{7.96}$$

is fixed by Bethe ansatz equations. The asymptotics of $\mathsf{W}^{(3)}$ is given by

$$\log\left(\mathsf{W}^{(as)}_3(\mathrm{i}\,\mathsf{e}^{\theta})\right) = r\sinh^2\left(\tfrac{1}{2}\,\theta\right) - \left(\frac{2\,p_1}{a_1}+\frac{2\,p_2}{a_2}\right)\theta \tag{7.97}$$

Finally, we compute the integrals over τ in terms containing $G^{(\pm)}_{ij}$ and combine terms so that integral representation can be written as

$$\begin{aligned}\log \mathsf{W}^{(i)} = \log \mathsf{W}^{(i)}_{as} &+ \sum_{j=1,3}\int_{-\infty}^{\infty} F^{(+)}_{ij}(\theta-\tau)\log(1+\mathsf{e}^{-\mathrm{i}\epsilon^{(j)}(\tau)})\mathsf{d}\tau \\ &- \sum_{j=1,3}\int_{-\infty}^{\infty} F^{(-)}_{ij}(\theta-\tau)\log(1+\mathsf{e}^{\mathrm{i}(\epsilon^{(j)}(\tau))^*})\mathsf{d}\tau\end{aligned} \tag{7.98}$$

The Fourier transform of kernel $F^{(\pm)}$ can be computed as

$$\begin{aligned}
\hat{F}^{(\pm)}_{11}(\nu) &= +\frac{1}{2\sinh(\pi\nu)}\,\Big[\big(\hat{G}_{11}(\nu)+1\big)\,\mathsf{e}^{\pm\pi\nu}+\hat{G}_{12}(\nu)\Big]\\
\hat{F}^{(\pm)}_{13}(\nu) &= +\frac{1}{2\sinh(\pi\nu)}\,\hat{G}_{33}(\nu)\\
\hat{F}^{(\pm)}_{21}(\nu) &= -\frac{1}{2\sinh(\pi\nu)}\,\Big[\big(\hat{G}_{21}(\nu)+\mathsf{e}^{\frac{\pi\delta\nu}{2}}\,\hat{G}_{31}(\nu)\big)\,\mathsf{e}^{\pm\pi\nu}+\hat{G}_{22}(\nu)+1+\mathsf{e}^{\frac{\pi\delta\nu}{2}}\,\hat{G}_{32}(\nu)\Big]\\
\hat{F}^{(\pm)}_{23}(\nu) &= -\frac{1}{2\sinh(\pi\nu)}\,\Big[\hat{G}_{23}(\nu)+\mathsf{e}^{\frac{\pi\delta\nu}{2}}\big(\hat{G}_{33}(\nu)+1\big)\Big] \qquad (7.99)\\
\hat{F}^{(\pm)}_{31}(\nu) &= -\frac{\cosh(\frac{\pi\nu}{2})}{\sinh(\pi\nu)}\,\Big[\hat{G}_{31}(\nu)\,\mathsf{e}^{\pm\pi\nu}+\hat{G}_{32}(\nu)\Big],\\
\hat{F}^{(\pm)}_{33}(\nu) &= -\frac{\cosh(\frac{\pi\nu}{2})}{\sinh(\pi\nu)}\,\Big[\hat{G}_{33}(\nu)+1\Big].
\end{aligned}$$

and final answer reads

$$\begin{aligned}
\hat{F}^{(\pm)}_{11}(\nu) &= +\frac{\mathsf{e}^{\pm\frac{1}{2}\pi(1+a_2)\nu}}{4\cosh(\frac{\pi\nu}{2})\sinh(\frac{\pi a_2\nu}{2})}, & \hat{F}^{(\pm)}_{13}(\nu) &= -\frac{1}{4\cosh(\frac{\pi\nu}{2})\sinh(\frac{\pi a_2\nu}{2})}\\
\hat{F}^{(\pm)}_{21}(\nu) &= -\frac{\mathsf{e}^{\pm\frac{1}{2}\pi(1-a_1)\nu}}{4\cosh(\frac{\pi\nu}{2})\sinh(\frac{\pi a_1\nu}{2})}, & \hat{F}^{(\pm)}_{23}(\nu) &= -\frac{1}{4\cosh(\frac{\pi\nu}{2})\sinh(\frac{\pi a_1\nu}{2})}\\
\hat{F}^{(\pm)}_{31}(\nu) &= -\frac{\mathsf{e}^{\pm\frac{\pi\nu}{2}}\,\sinh(\frac{\pi\delta\nu}{2})}{2\sinh(\frac{\pi a_1\nu}{2})\,\sinh(\frac{\pi a_2\nu}{2})}, & \hat{F}^{(\pm)}_{33}(\nu) &= -\frac{\sinh(\frac{\pi\nu}{2})}{2\sinh(\frac{\pi a_1\nu}{2})\,\sinh(\frac{\pi a_2\nu}{2})}.
\end{aligned} \qquad (7.100)$$

Similarly, any other some over Bethe roots can be computed. In particular, energy is given by

$$E^{(1)} = -M\cos\frac{\pi\delta}{2}\sum\cosh\theta^{(1)}_k \qquad (7.101)$$

Integral representation will be

$$\begin{aligned}
RE^{(1)} &= -r^2\cos^2\frac{\pi\delta}{2}\int_{-\Theta}^{\Theta}\sinh\tau\,\mathsf{z}^{(1)}(\tau)\mathsf{d}\tau\\
&+\int_{-\infty}^{\infty}\sum_{k=1,3}\Big(H^{(+)}_{ik}(\tau)b^{(k)}(\tau)-H^{(-)}_{ik}(\tau)(b^{(k)}(\tau))^*\Big)\,\mathsf{d}\tau
\end{aligned} \qquad (7.102)$$

As expected, there is quadratic divergence: first term grows as $\mathsf{e}^{2\Theta}\sim\Lambda^2$ in large Θ limit, similarly to what we've seen in perturbation theory. However, this term is proportional to R^2, hence the divergence is absorbed into infinite shift of vacuum energy level at $R = \infty$. The resulting expression for scaling function thus has well-defined limit $\Theta \to \infty$, $r = 4N\mathsf{e}^{\Theta} = const$.

$$\begin{aligned} E_N^{(i)} &= N\,\varepsilon_\infty^{(i)} + \frac{1}{2\pi}\sum_{k=1,3}\int_{-\infty}^{\infty}\mathsf{d}\theta\, H_{ik}^{(+)}(-\theta + \mathrm{i}\omega_k)\ \log\left(1 + \mathsf{e}^{-\mathrm{i}\varepsilon_k(\theta)}\right) \\ &\quad -\frac{1}{2\pi}\sum_{k=1,3}\int_{-\infty}^{\infty}\mathsf{d}\theta\, H_{ik}^{(-)}(-\theta - \mathrm{i}\omega_k)\ \log\left(1 + \mathsf{e}^{-\mathrm{i}\varepsilon_k^*(\theta)}\right), \end{aligned} \tag{7.103}$$

where

$$\begin{aligned} \hat{H}_{11}^{(\pm)}(\nu) &= +\mathsf{e}^{\pm\frac{1}{2}\pi(1+a_2)\nu}\hat{H}_{13}^{(\pm)}(\nu)\,, &\qquad \hat{H}_{13}^{(\pm)}(\nu) &= -\frac{2\sin(\Theta\nu)}{\cosh(\frac{\pi\nu}{2})} \\ \hat{H}_{21}^{(\pm)}(\nu) &= -\mathsf{e}^{\pm\frac{1}{2}\pi(1-a_1)\nu}\hat{H}_{13}^{(\pm)}(\nu)\,, &\qquad \hat{H}_{23}^{(\pm)}(\nu) &= \hat{H}_{13}^{(\pm)}(\nu)\,. \end{aligned}$$

7.3 Asymptotic expansion of connection coefficients

Integral representations (7.98),(7.100) for connection coefficients allow to complete the asymptotical expansions (6.136), (6.130), (7.95). Indeed, closing the contour of integration in Fourier transform we can rewrite kernels $F_{ik}^{(\pm)}$ as a sum over its poles:

$$F_{ik}^{(\pm)}(\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathsf{e}^{\mathrm{i}\theta\nu}\hat{F}_{ik}^{(\pm)}(\nu)\mathsf{d}\nu = \mathrm{i}\sum_{\nu_j^{(ik)}}\mathrm{Res}\hat{F}|_{\nu=\nu_j^{(ik)}}\mathsf{e}^{\mathrm{i}\nu_j^{(ik)}\theta} \tag{7.104}$$

where points $\nu_j^{(ik)}$ are poles of $\hat{F}_{(ik)}^{(\pm)}$ located in upper half plane. They are arranged in three series

$$\nu_j^{(a_1)} = \frac{2\mathrm{i}j}{a_1} \qquad \nu_j^{(a_2)} = \frac{2\mathrm{i}j}{a_2} \qquad \nu_j^{(1)} = \mathrm{i}(2j-1)\,, \qquad j \in \mathbb{N} \tag{7.105}$$

Consequently we may write the asymptotic expansion for connection coefficients for conformal problem as

$$\log\left(D^{(1)}(\lambda)\right) = -\left(\gamma_E + \psi(\tfrac{1}{2}a_2)\right)\lambda + \sum_{n=1}^{\infty}\mathcal{J}_n\lambda^{1-2n} + \sum_{n=1}^{\infty}\mathcal{H}_n\lambda^{\frac{2}{a_2}(1-2n)} \tag{7.106}$$

$$\log\left(D^{(2)}(\lambda)\right) = -\left(\gamma_E + \psi(\tfrac{1}{2}a_1)\right)\lambda + \sum_{n=1}^{\infty}\mathcal{J}_n\lambda^{1-2n} + \sum_{n=1}^{\infty}\mathcal{H}_n\lambda^{\frac{2}{a_1}(1-2n)} \tag{7.107}$$

$$\log\left(D^{(3)}(\lambda)\right) = \frac{\pi\lambda}{\cos(\frac{1}{2}\pi\delta)} + \sum_{n=1}^{\infty}\mathcal{H}_n\lambda^{\frac{2}{a_1}(1-2n)} + \sum_{n=1}^{\infty}H_n\lambda^{\frac{2}{a_2}(1-2n)} \tag{7.108}$$

with

$$\mathcal{J}_{2n-1} = \frac{(-1)^{n+1}}{\pi\cos\left((n-\frac{1}{2})\pi\delta\right)}\mathrm{Im}\left((-1)^n \mathrm{i}\mathsf{e}^{\frac{\pi a_2}{2}(2n-1)} g_1(\mathrm{i}(2n-1)) + g_3(\mathrm{i}(2n-1))\right) + B_{2n-1} \tag{7.109}$$

$$\mathcal{H}_n^{(i)} = \frac{1}{\pi a_j \cos\left(\frac{\pi n}{a_j}\right)}\mathrm{Im}\left(\mathsf{e}^{\frac{2\pi n\mathrm{i}}{a_j}} g_1\left(\frac{2\mathrm{i}}{a_i}(2n-1)\right) - (-1)^n g_3\left(\frac{2\mathrm{i}}{a_i}(2n-1)\right)\right) \tag{7.110}$$

and B_n are Bernoulli numbers, which appear because Wronskians $\mathtt{W}^{(1)}$, $\mathtt{W}^{(2)}$ are normalized by gamma function. Similarly, for massive problem we obtain

$$\log\left(\mathsf{W}_i(\mathsf{e}^\theta)\right)\Big|_{\theta\to+\infty} \asymp -4\,\rho\,\theta\sinh(\theta) + 2\rho\, C_{-1}^{(i)}\cosh(\theta) - k_j\theta - \tfrac{1}{2}\log\left(\sin(\pi k_j)\right)$$
$$-\sum_{n=1}^{\infty}\frac{\mathsf{e}^{-(2n-1)\theta}\, I_{2n-1}}{\left(M\cos(\frac{\pi\delta}{2})\right)^{2n-1}} + \sum_{n=0}^{\infty}\frac{(-1)^n \mathsf{e}^{-2n\theta/a_j}\, H_n^{(j)}}{2\cos(\frac{n\pi}{a_j})\left(M\cos(\frac{\pi\delta}{2})\right)^{2n/a_j}} \tag{7.111a}$$

It is also possible to write asymptotical expansion for $\theta \to -\infty$:

$$\log\left(\mathsf{W}_i(\mathsf{e}^\theta)\right)\Big|_{\theta\to-\infty} \asymp -4\,\rho\,\theta\sinh(\theta) + 2\rho\, C_{-1}^{(i)}\cosh(\theta) - k_j\theta - \tfrac{1}{2}\log\left(\sin(\pi k_j)\right)$$
$$-\sum_{n=1}^{\infty}\frac{\mathsf{e}^{+(2n-1)\theta}\, \bar{I}_{2n-1}}{\left(M\cos(\frac{\pi\delta}{2})\right)^{2n-1}} + \sum_{n=0}^{\infty}\frac{(-1)^n \mathsf{e}^{+2n\theta/a_j}\, \bar{H}_n^{(j)}}{2\cos(\frac{n\pi}{a_j})\left(M\cos(\frac{\pi\delta}{2})\right)^{2n/a_j}}\,, \tag{7.111b}$$

Asymptotic expression for W_3 can be written as

$$\log\left(\mathsf{W}_3(\pm\mathrm{i}\,\mathsf{e}^\theta)\right)\Big|_{\theta\to+\infty} \asymp \frac{2\pi\rho\cosh(\theta)}{\cos(\frac{\pi\delta}{2})}$$
$$-\left(k_1+k_2\right)\theta - \tfrac{1}{2}\log\left(\sin(\pi k_1)\right) - \tfrac{1}{2}\log\left(\sin(\pi k_2)\right)$$
$$+\sum_{n=0}^{\infty}\frac{\mathsf{e}^{-2n\theta/a_1}\, H_n^{(1)}}{2\cos(\frac{n\pi}{a_1})\left(M\cos\left(\frac{\pi\delta}{2}\right)\right)^{2n/a_1}} + \sum_{n=0}^{\infty}\frac{\mathsf{e}^{-2n\theta/a_2}\, H_n^{(2)}}{2\cos(\frac{n\pi}{a_2})\left(M\cos(\frac{\pi\delta}{2})\right)^{2n/a_2}} \tag{7.112}$$

In order to find asymptotics of W_3 for $\theta \to -\infty$ one needs to replace H_n by $\bar{H}_n$ in (7.112). The integrals appearing in (7.112), (7.111a), (7.111b) are given by

$$I_{2n-1} = \bar{I}_{2n-1} = \pm\frac{\left(M\cos(\frac{\pi\delta}{2})\right)^{2n-1}}{\cos\left(\frac{\pi\delta}{2}(2n-1)\right)}\mathrm{Im}\Big(L_+\big(\pm\mathrm{i}(2n-1)\big) + \mathsf{e}^{\mp\frac{\mathrm{i}\pi}{2}(2n-1)a_1} L_-\big(\pm\mathrm{i}(2n-1)\big)\Big) \tag{7.113}$$

$$\begin{aligned} H_n^{(i)} &= +\frac{2}{a_i}\left(M\cos(\tfrac{\pi\delta}{2})\right)^{\frac{2n}{a_i}}\mathrm{Im}\Big(L_+\big(+\tfrac{2\mathrm{i}n}{a_i}\big) + (-1)^{i-1}L_-\big(+\tfrac{2\mathrm{i}n}{a_i}\big)\Big) \\ \bar{H}_n^{(i)} &= -\frac{2}{a_i}\left(M\cos(\tfrac{\pi\delta}{2})\right)^{\frac{2n}{a_i}}\mathrm{Im}\Big(L_+\big(-\tfrac{2\mathrm{i}n}{a_i}\big) + (-1)^{i-1}L_-\big(-\tfrac{2\mathrm{i}n}{a_i}\big)\Big)\,. \end{aligned} \tag{7.114}$$

7.4 Short form of integral equations

First we note that all poles of $\log(1+f_{12}^{-1}(\lambda))$ lie between rays $\mathrm{Arg}\lambda = \mathrm{i}\omega_2$ and $\mathrm{Arg}\lambda = -\mathrm{i}\omega_2$. It means that function b_2 admits analytical continuation to b_1. More precisely,

$$\int_{-\infty}^{\infty} B(\theta - \mathrm{i}\omega_2) b_2(\theta) \mathsf{d}\theta = \int_{-\infty}^{\infty} B(\theta - \mathrm{i}\omega_1) b_1(\theta) \mathsf{d}\theta \tag{7.115}$$

provided $B(\theta)$ is regular function in the strip $\mathrm{Im}\,\theta \in [-\omega_1 : -\omega_2]$. In particular if we introduce

$$g_k(\nu) = \mathsf{e}^{-\nu\omega_k} \int_{-\infty}^{\infty} \mathsf{e}^{-\mathrm{i}\nu\theta} \left(\log(1+\mathsf{e}^{-\mathrm{i}\epsilon^{(k)}(\theta)}) - \log(1+\mathsf{e}^{-\mathrm{i}\epsilon^{(k)}(-\infty)})\right) \mathsf{d}\theta \tag{7.116}$$

then

$$g_1(\nu) = g_2(\nu) \tag{7.117}$$

This observation allows to expel the contour $\mathcal{C}_1$ from the system of NLIEs completely. In (7.58) we replace all integrals involving $\epsilon^{(2)}$ by similar integrals with $\epsilon^{(1)}$:

$$\int_{-\infty}^{\infty} G_{i1}^{(+)}(\theta-\theta') \log(1+\mathsf{e}^{-\mathrm{i}\epsilon^{(1)}(\theta')}) \mathsf{d}\theta' = \int_{-\infty}^{\infty} \mathsf{d}\theta' \int_{-\infty}^{\infty} \mathsf{d}\nu \mathsf{e}^{\mathrm{i}\nu(\theta-\theta')} \hat{G}_{i2}(\nu) \log(1+\mathsf{e}^{-\mathrm{i}\epsilon^{(2)}(\theta')}) \tag{7.118}$$

Then we don't need equation for $\epsilon^{(1)}$ anymore, as r.h.s. of the system does not involve it. Introducing notations

$$\epsilon_+ = \epsilon_+^{(3)}\,, \qquad \epsilon_- = \epsilon_+^{(2)}\,, \qquad \chi_+ = 0\,, \qquad \chi_- = \tfrac{1}{2}\pi a_1 \tag{7.119}$$

and setting $\omega_2 = \frac{1}{2}\pi$ we we can write the answer as

$$\begin{aligned} \varepsilon_\sigma(\theta) &= r\,\sinh(\theta - \mathrm{i}\chi_\sigma) - 2\pi k_\sigma \\ &+ \sum_{\sigma'=\pm} \int_{-\infty}^{\infty} \frac{\mathsf{d}\theta'}{\pi}\, G_{\sigma\sigma'}(\theta-\theta')\ \mathrm{Im}\Big[\log\left(1+e^{-\mathrm{i}\varepsilon_{\sigma'}(\theta'-\mathrm{i}0)}\right)\Big]\,. \end{aligned} \tag{7.120}$$

From the reasoning above and expressions for $\hat{G}_{ik}$ obtained earlier it follows that kernels $G_{\sigma,\sigma'}$ can be computed as

$$\hat{G}_{--}(\nu) = \hat{G}_{22}(\nu)(1+\mathsf{e}^{\pi\nu}) \tag{7.121}$$

$$\hat{G}_{-+}(\nu) = \hat{G}_{23}(\nu) \tag{7.122}$$

$$\hat{G}_{+-}(\nu) = \hat{G}_{32}(\nu)(1+\mathsf{e}^{\pi\nu}) \tag{7.123}$$

$$\hat{G}_{++}(\nu) = \hat{G}_{33}(\nu) \tag{7.124}$$

The result can be represented as

$$G_{\pm\pm}(\theta) = G_{a_1}(\theta) + G_{a_2}(\theta)\ , \qquad G_{\pm\mp}(\theta) = \hat{G}_{a_1}(\theta) - \hat{G}_{a_2}(\theta) \tag{7.125}$$

$$\begin{aligned} G_a(\theta) &= \int_{-\infty}^{\infty} \mathsf{d}\nu\, \frac{\mathsf{e}^{\mathrm{i}\nu\theta}\ \sinh(\frac{\pi\nu}{2}(1-a))}{2\cosh(\frac{\pi\nu}{2})\sinh(\frac{\pi\nu a}{2})} \\ \hat{G}_a(\theta) &= \int_{-\infty}^{\infty} \mathsf{d}\nu\, \frac{\mathsf{e}^{\mathrm{i}\nu\theta}\ \sinh(\frac{\pi\nu}{2})}{2\cosh(\frac{\pi\nu}{2})\sinh(\frac{\pi\nu a}{2})}\ . \end{aligned} \tag{7.126}$$

7.5 Numerical solution of integral equations

To compute functions $\epsilon^{(i)}$ numerically we used the following algorithm:

- Define three arrays of $N+1$ complex numbers each representing the values of functions $\epsilon^{(i)}$:
$$\epsilon_k^{(i)} = \epsilon^{(i)}(-\Theta + hk) \tag{7.127}$$

- Define 18 arrays of $2N+1$ complex numbers each representing elements $G^{(\pm)}_{ik,l}(-2\Theta + hl)$. Compute Fourier transform of kernels G on this lattice.

- Initialize $\epsilon_k^{(i)}$ with asymptotic values (7.45) for conformal case or (7.66) for massive case respectively
$$\epsilon_{k,0}^{(i)} = 2\pi z_i(-\Theta + hk) \tag{7.128}$$

- Repeat till converges

$$\begin{aligned} \epsilon^{(i)}_{k,j+1} &= \alpha\epsilon^{(i)}_{k,j} + (1-\alpha)\left[2\pi z_i(-\Theta+hk) - \mathrm{i}h\left(\Delta^{(i)}_{k,j} - \bar{\Delta}^{(i)}_{k,j}\right)\right] \\ \Delta^{(i)}_{k,j} &= \sum_{l=2}^{N}\sum_{m=1}^{3}\left(G^{(+)}_{im,k-l+N+1} b^{(m)}_{l-1,j} + 4G^{(+)}_{im,k-l+N} b^{(m)}_{l,j} + G^{(+)}_{im,k-l+N-1} b^{(m)}_{l+1,j}\right) \\ \bar{\Delta}^{(i)}_{k,j} &= \sum_{l=2}^{N}\sum_{m=1}^{3}\left(G^{(-)}_{im,k-l+N+1} \bar{b}^{(m)}_{l-1,j} + 4G^{(-)}_{im,k-l+N} \bar{b}^{(m)}_{l,j}) + G^{(-)}_{im,k-l+N-1} \bar{b}^{(m)}_{l+1,j}\right) \\ b^{(i)}_{k,j} &= \log\left(1 + \exp(-\mathrm{i}\epsilon^{(i)}_{k,j})\right) \\ \bar{b}^{(i)}_{k,j} &= \log\left(1 + \exp\left(+\mathrm{i}\left(\epsilon^{(i)}_{k,j}\right)^*\right)\right) \end{aligned} \tag{7.129}$$

where parameter α is demphering parameter such that $\alpha \in (0:1)$. In practice depending on values of parameters of the model optimal α varied between 0.5 and 0.9.

- Use

$$d_j = \sum_{i,k} |\epsilon^{(i)}_{k,j+1} - \epsilon^{(i)}_{k,j}| \tag{7.130}$$

  as a measure of convergence. For wide range of parameters r and δ and $N = 4000$ 50 iterations were sufficient to reduce d_j below 10^{-14}, i.e. close to the machine precision.

For conformal case the limit of functions $\epsilon^{(i)}$ at $\theta \to -\infty$ is nonzero, so $k < k_{min}$ on each iteration we replace values of $\epsilon^{(i)}_k$ by analytically precomputed limiting value. Same modification has to be performed for lattice regularized case at both infinities. Then the values of integrals of motion, in particular scaling function, can be computed using known integral representations.

We performed said computation for conformal, massive and lattice-regularized cases. The results are in agreement with both asymptotical expressions (see Fig.7.5) and results of straightforward numerical solution of regularized Bethe ansatz equations (Fig.7.6) . This observation is further evidence in support of the statement that the ODE/IQFT construction provides an exact description of vacuum state of the Bukhvostov-Lipatov model, i.e. the physical vacuum of the model is filled with quasiparticles with rapidities given precisely by zeroes of corresponding connection coefficients of the auxillary linear problem.

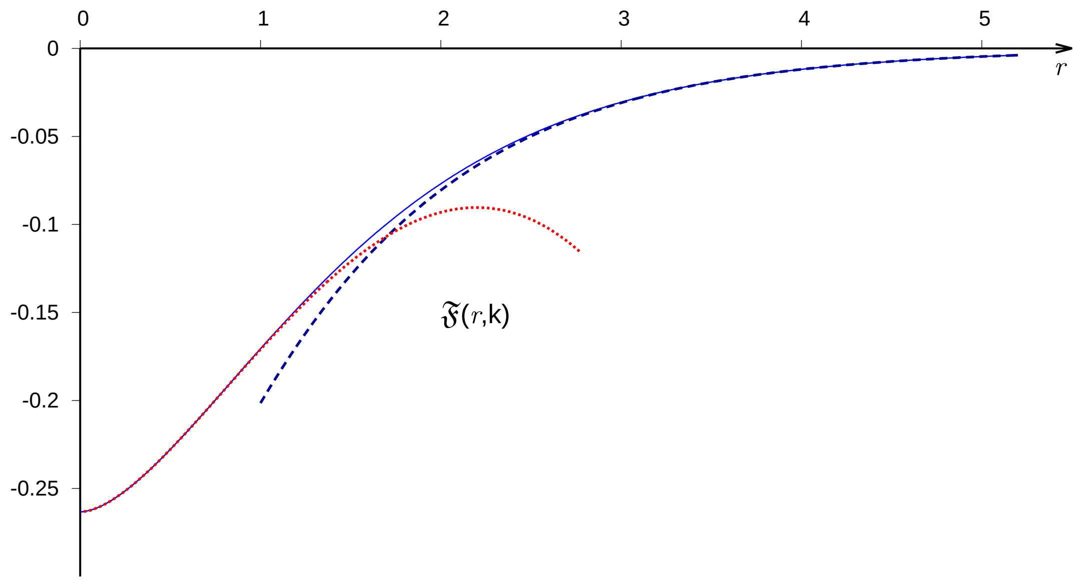


Figure 7.5: Scaling function of Bukhvostov-Lipatov model at $\delta = \frac{17}{47}$, $p_1 = \frac{1}{10}$, $p_2 = \frac{1}{20}$. Solid line represents the value computed from solving NLIEs. Dashed lines represent UV and IR asymptotics obtained from conformal perturbation theory and renormalized perturbation theory respectively.

7.6 Vacuum energy from solution of modified sinh-Gordon equation

Consider Fateev model in unitary regime (2.54). As discussed in [77], its ODE dual is given by (6.1),(6.2) with $a_3 \neq 0$. The Wilson loop

$$W = \mathrm{Tr}\left(\mathcal{P}\exp\int_C \boldsymbol{A}\right) \tag{7.131}$$

generates infinitely many integrals of motion of MShG equation. The contour C in (7.131) is Pochammer loop - the only non-contractible loop that winds around each of the singularities. We define charges $\mathfrak{q}_i$, $\bar{\mathfrak{q}}_i$ through its asymptotic expansions:

$$\log W(\theta) \sim -\mathfrak{q}_0 \mathsf{e}^{\theta} + \sum_{n=1}^{\infty} c_n \mathfrak{q}_{2n-1} \mathsf{e}^{-(2n-1)\theta}, \qquad \theta \to +\infty \tag{7.132}$$

$$\log W(\theta) \sim -\bar{\mathfrak{q}}_0 \mathsf{e}^{-\theta} + \sum_{n=1}^{\infty} c_n \bar{\mathfrak{q}}_{2n-1} \mathsf{e}^{(2n-1)\theta}, \qquad \theta \to -\infty \tag{7.133}$$

In the work [74] the following expression for $\mathfrak{q}_1$ was obtained

$$\mathfrak{q}_1 = \mathsf{e}^{\frac{\mathrm{i}\pi(a_1+a_2)}{2}} \oint_{C_w} T\mathsf{d}w + \Theta \mathsf{d}\bar{w} \tag{7.134}$$

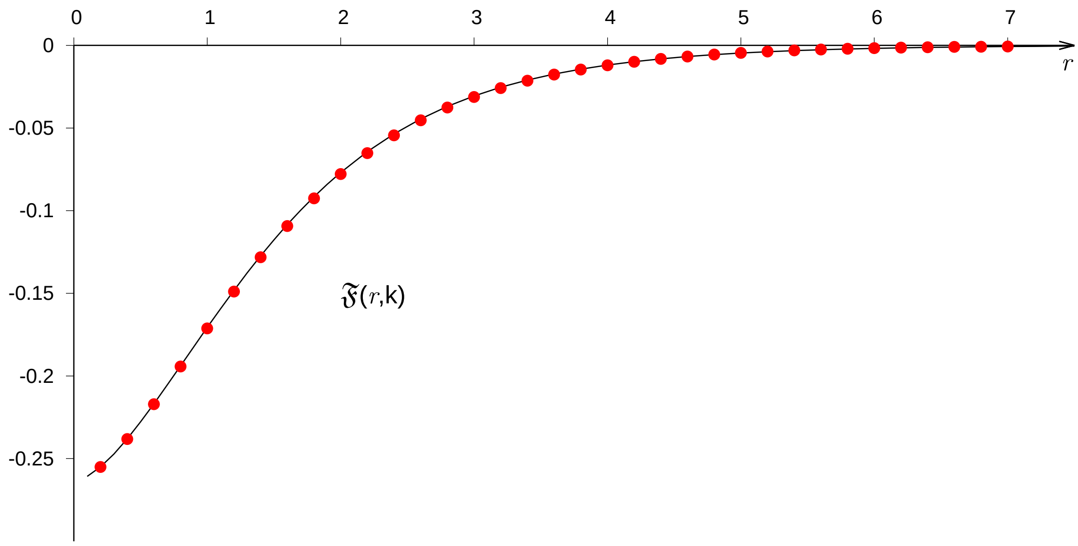


Figure 7.6: Scaling function of Bukhvostov-Lipatov model at $\delta = \frac{17}{47}$, $p_1 = \frac{1}{10}$, $p_2 = \frac{1}{20}$. Solid line represents the answer from NLIEs, dots stand for values computed within lattice-regularized Bethe ansatz framework at $N = 500$.

where functions T, $\bar{T}$ and Θ are related to stress-energy tensor of classical sinh-Gordon theory,

$$T = (\partial_w \hat{\eta})^2, \qquad \Theta = 4\sinh^2 \hat{\eta}, \qquad \bar{T} = (\partial_{\bar{w}} \hat{\eta})^2 \tag{7.135}$$

function $\hat{\eta}(w, \bar{w})$ is a solution of sinh-Gordon equation, and image C_w of Pochammer loop C is a union of oriented line segments

$$C_w = \overrightarrow{w_0 b} \cup \overrightarrow{\bar{b}\bar{c}} \cup \overrightarrow{cd} \cup \overrightarrow{\bar{d}\bar{a}} \cup \overrightarrow{\bar{a}w_0} \tag{7.136}$$

where points a, c belong to $[w_2, w_3]$ and points b,d to $[w_1, w_3]$ Sinh-Gordon equation implies that there exists function $\Phi(w, \bar{w})$ such that

$$\partial_w^2 \Phi = T, \qquad \partial_w \partial_{\bar{w}} \Phi = \Theta, \qquad \partial_{\bar{w}}^2 \Phi = \bar{T} \tag{7.137}$$

It allows to simplify equation (7.134) as follows

$$\mathfrak{q}_1 = \mathrm{e}^{\frac{\mathrm{i}\pi(a_1+a_2)}{2}} \oint d\partial_w \Phi \tag{7.138}$$

Since Wilson loop (7.131) is invariant under continuous deformations of contour C, the following quantities must be constants

$$\Delta_1 = 2\mathrm{i}(\mathrm{e}^{\frac{\mathrm{i}\pi a_1}{2}} \partial_w \Phi(b) - \mathrm{e}^{-\frac{\mathrm{i}\pi a_1}{2}} \partial_w \Phi(\bar{b})) = const, \qquad b \in [w_1, w_3] \tag{7.139}$$

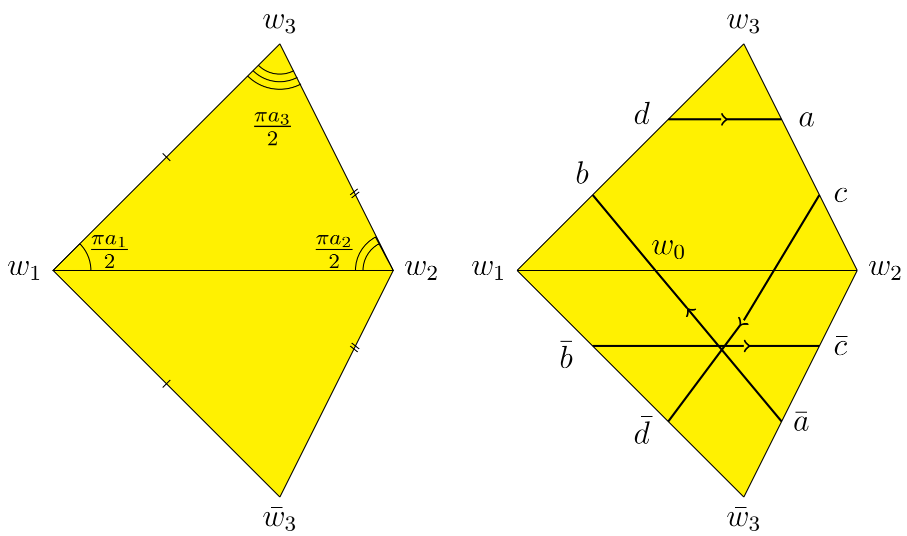


Figure 7.7: Domain $\mathbb{D}_F$. For the case $a_3 > 0$ it is obtained by identifying sides $[w_1, w_3]$ and $[w_2, w_3]$ with $[w_1, \bar{w}_3]$ and $[w_2, \bar{w}_3]$ repsectivly. The right plate shows the image C_w of the Pochammer loop C under Schwarz-Christoffel map.

$$\Delta_2 = 2\mathrm{i}(\mathsf{e}^{-\frac{\mathrm{i}\pi a_2}{2}}\partial_w\Phi(c) - \mathsf{e}^{\frac{\mathrm{i}\pi a_2}{2}}\partial_w\Phi(\bar{c})) = const, \qquad c \in [w_2, w_3] \tag{7.140}$$

The final answer for $\mathfrak{q}_1$ therefore reads:

$$\mathfrak{q}_1 = \left(\Delta_1 \sin\frac{\pi a_2}{2} + \Delta_2 \sin\frac{\pi a_1}{2}\right) \tag{7.141}$$

Making use of (7.135), (7.137) and applying Green's identity one can relate those constants to area integral over domain D (which is a union of triangle (w_1, w_2, w_3) and its mirror image $(w_1, w_2, \bar{w}_3)$).

$$4\int_D \mathsf{d}^2 w\ \sinh^2\hat{\eta} = -\frac{1}{4}\left(|w_1 - w_3|\Delta_1 + |w_2 - w_3|\Delta_2\right) + \sum_{i=1}^{3}\frac{\pi a_i}{2}\left(\frac{2p_i}{a_i} - \frac{1}{2}\right)^2 \tag{7.142}$$

In order to take limit $a_3 \to 0$ it is convenient to rewrite it as follows:

$$4\int_D \mathsf{d}^2 w\ \sinh^2\hat{\eta} = -\frac{|w_1 - w_2|}{8\sin\frac{\pi a_3}{2}}\left(\mathfrak{q}_1 + \bar{\mathfrak{q}}_1\right) + \sum_{i=1}^{3}\frac{\pi a_i}{2}\left(\frac{2p_i}{a_i} - \frac{1}{2}\right)^2 \tag{7.143}$$

Work [77] also relates $\mathfrak{q}_1$ to the coefficient I_1 in asymptotic expansion of connection coefficients:

$$\mathfrak{q}_1 = \frac{8\pi^2 I_1}{\rho\Gamma(\frac{a_1}{2})\Gamma(\frac{a_2}{2})\Gamma(\frac{a_3}{2})} \tag{7.144}$$

Therefore the quantum integral of motion I_1 admits the following expansion:

$$4\int_D \mathsf{d}^2 w\ \sinh^2\hat{\eta} - \sum_{i=1}^{3}\frac{\pi a_i}{2}\left(\frac{2p_i}{a_i} - \frac{1}{2}\right)^2 = -\frac{2\pi\Gamma(1-\frac{a_3}{2})I_1}{\Gamma(\frac{a_1}{2})\Gamma(\frac{a_2}{2})}|w_1 - w_2| \tag{7.145}$$

In the limit $a_3 = 0$ for conformal case we obtain:

$$4 \int\limits_D \mathsf{d}^2 w \ \sinh^2 \hat{\eta} - \sum_{i=1}^{2} \frac{\pi a_i}{2} \left(\frac{2p_i}{a_i} - \frac{1}{2} \right)^2 = -2\pi \mathcal{J}_1 \tag{7.146}$$

and for massive case similar expression holds:

$$I_1 = \frac{1}{4\rho} \left(-\frac{8}{\pi} \int_{\mathbb{D}_{\mathrm{BL}}} \mathsf{d}^2 w \sinh^2(\hat{\eta}) + 2 \sum_{i=1} a_i \left(|k_i| - \tfrac{1}{2} \right)^2 \right) , \tag{7.147}$$

The exact expression for scaling function can be obtained via relations

$$\mathfrak{F} = \frac{R}{2\pi} (I_1 + \bar{I}_1) \tag{7.148}$$

$$\mathcal{J}_1 = \mathfrak{F} + \mathfrak{f}_{FB} \tag{7.149}$$

for massive and conformal cases respectively, which follow from integral representations discussed previously.

Conclusion

In this work we investigated the vacuum state of Bukhvostov-Lipatov model in finite volume with quasiperiodic boundary conditions. We have computed long distance and short distance behaviour of scaling function using perturbation theory. We have extended the treatment of Bethe ansatz equations for Bukvostov-Lipatov model beyond the scope of the original work [48]. We have introduced lattice-type regularization into the model which enabled numerical solutions. We have studied the model numerically for various values of the parameters.

In this work we have established the ODE/IM correspondence between Bukhvostov-Lipatov model and modified sinh-Gordon equation. This work fills the gap in studies of Fateev model. In doing so we have discovered new type of correspondence between quantum and classical systems: an exact duality between particular solution of an integrable PDE and the eigenstate of an integrable QFT. Using this correspondence we have derived Destri - De Vega type NLIE equivalent to the full system of Bethe ansatz equations, which is well suited for numerical solution and does not require a regularization. We have also proved an elegant formula expressing vacuum energy via a special solution of modified sinh-Gordon model. All approaches to computing vacuum energy are in good agreement.

This topic definitely merits further research. One possible direction is to extend this method to the most interesting regime $a_1, a_2 > 0$, $a_3 < 0$ of Fateev model which corresponds to integrable deformations of $O(4)$ sigma model. The difficulty presented by this generalization is the fact that the NLIE of the type presented in this work can not be applied to this case. One also hopes to apply this approach to sigma models arising in the context of integrable supersymmetric gauge theories in higher dimension, most notably AdS_5 sigma model which appears in context of $\mathcal{N} = 4$ SYM theory.

Another interesting question is to construct the lattice model which is exactly described by "lattice-regularized" BAE. There are some indication that Osp(2|2) supersymmetric quantum spin chain discussed in [93, 94] can be this model, but so far it was not proved.

Finally, it would be interesting to extend this work to excited states. In principle the ODE/IM construction was applied to the excited states of Fateev model in unitary regime [77]. The open question is to construct the excited states of Bukhvostov-Lipatov model.

Derivation of functional relations

In this appendix we present the derivation of functional relations (6.99a-6.99g)(massive case) and (6.70-6.73)(conformal case) for the connection coefficients, starting from the main functional relation (6.44) for the connection matrices,

$$\boldsymbol{S}^{(i,k)}(\lambda)\,\mathsf{e}^{-2\pi\mathrm{i}p_k(\lambda)\sigma_3}\,\boldsymbol{S}^{(k,j)}(\lambda q_k^{-1})\,\mathsf{e}^{-2\pi\mathrm{i}p_j(\lambda q_k^{-1})\,\sigma_3}\,\boldsymbol{S}^{(j,i)}(\lambda q_i)\,\mathsf{e}^{-2\pi\mathrm{i}p_i(\lambda q_i)\,\sigma_3}=-\boldsymbol{I}\,. \quad (\mathrm{A.1})$$

Here for massive case we should substitute

$$p_1(\lambda)=p_1\,,\qquad p_2(\lambda)=p_2\,,\qquad p_3(\lambda)=\rho(\lambda-\lambda^{-1}) \quad (\mathrm{A.2})$$

while for conformal case slightly different notation is used

$$p_1(\lambda)=p_1\,,\qquad p_2(\lambda)=p_2\,,\qquad p_3(\lambda)=\lambda \quad (\mathrm{A.3})$$

Below we shall provide detailed derivation for the massive case only, but the functional relations governing conformal connection matrices can be obtained via substitution

$$\rho(\lambda-\lambda^{-1})\mapsto\lambda \quad (\mathrm{A.4})$$

in the following derivation.

Take $(i,j,k)=(1,2,3)$ and express $\boldsymbol{S}^{(1,3)}$ through $\boldsymbol{S}^{(1,2)}$ and $\boldsymbol{S}^{(2,3)}$ in two different ways: from the main relation (A.1)

$$\boldsymbol{S}^{(1,3)}_{\sigma,\sigma''}(\lambda)=-\sum_{s=\pm}\mathsf{e}^{-2\pi\mathrm{i}p_1\sigma-2\pi\mathrm{i}p_2 s-2\pi\mathrm{i}\rho\,(\lambda-\lambda^{-1})}\;\boldsymbol{S}^{(1,2)}_{\sigma,s}(\lambda\mathsf{e}^{-\mathrm{i}\pi\delta})\,\boldsymbol{S}^{(2,3)}_{s,\sigma''}(\lambda)$$

and from the simple properties (6.41) of the connection matrices,

$$\boldsymbol{S}^{(1,3)}_{\sigma,\sigma''}(\lambda\mathsf{e}^{-\mathrm{i}\pi\delta})=\sum_{s=\pm}\boldsymbol{S}^{(1,2)}_{\sigma,s}(\lambda\mathsf{e}^{-\mathrm{i}\pi\delta})\;\boldsymbol{S}^{(2,3)}_{s,\sigma''}(\lambda\mathsf{e}^{-\mathrm{i}\pi\delta})\;. \quad (\mathrm{A.5})$$

This system contains both $\boldsymbol{S}^{(1,2)}_{\sigma,+}(\lambda\mathsf{e}^{-\mathrm{i}\pi\delta})$ and $\boldsymbol{S}^{(1,2)}_{\sigma,-}(\lambda\mathsf{e}^{-\mathrm{i}\pi\delta})$. Choose $\sigma''=+$ in both equations, fix a particular sign $\sigma'=\pm$, and then exclude $\boldsymbol{S}^{(1,2)}_{\sigma,-\sigma'}(\lambda\mathsf{e}^{-\mathrm{i}\pi\delta})$ among the above two equations. In this way one obtains

$$\boldsymbol{S}^{(3,1)}_{-,-\sigma}(\lambda \mathsf{e}^{-\mathrm{i}\pi\delta})\ \boldsymbol{S}^{(2,3)}_{-\sigma',+}(\lambda) + \mathsf{e}^{2\pi\mathrm{i}(\sigma p_1 - \sigma' p_2) + 2\mathrm{i}\pi\rho\,(\lambda-\lambda^{-1})}\ \boldsymbol{S}^{(3,1)}_{-,-\sigma}(\lambda)\ \boldsymbol{S}^{(2,3)}_{-\sigma',+}(\lambda \mathsf{e}^{-\mathrm{i}\pi\delta})$$

$$= \sigma \boldsymbol{S}^{(1,2)}_{\sigma,\sigma'}(\lambda \mathsf{e}^{-\mathrm{i}\pi\delta})\ \Big(\boldsymbol{S}^{(2,3)}_{\sigma',+}(\lambda \mathsf{e}^{-\mathrm{i}\pi\delta})\ \boldsymbol{S}^{(2,3)}_{-\sigma',+}(\lambda)\ - \boldsymbol{S}^{(2,3)}_{\sigma',+}(\lambda)\ \boldsymbol{S}^{(2,3)}_{-\sigma',+}(\lambda \mathsf{e}^{-\mathrm{i}\pi\delta})\ \mathsf{e}^{-4\pi\mathrm{i}\sigma' p_2}\Big)$$

which is equivalent to (6.99g) upon the substitution (6.98).

Next, take main relation (A.1) with $(i,j,k) = (1,2,3)$ and express $\boldsymbol{S}^{(1,3)}$ therein in terms of product of $\boldsymbol{S}^{(1,2)}$ and $\boldsymbol{S}^{(2,3)}$ using relation similar to (A.5) with shifted spectral parameter $\lambda \mapsto \lambda \mathsf{e}^{\frac{\mathrm{i}\pi\delta}{2}}$. It follows then

$$\sum_{\sigma''=\pm} \mathsf{e}^{2\pi\mathrm{i}\sigma''\rho(\lambda-\lambda^{-1})}\ \boldsymbol{S}^{(2,3)}_{\sigma,\sigma''}(\lambda)\ \boldsymbol{S}^{(3,2)}_{\sigma'',\sigma'}(\lambda) = -\sum_{\sigma''=\pm} \mathsf{e}^{-2\pi\mathrm{i}p_1\sigma'' - 2\pi\mathrm{i}p_2\sigma}\ \boldsymbol{S}^{(2,1)}_{\sigma,\sigma''}(\lambda)\ \boldsymbol{S}^{(1,2)}_{\sigma'',\sigma'}(\lambda \mathsf{e}^{-\mathrm{i}\pi\delta})\,.$$

Setting $\sigma = \sigma'$ and combining like terms one obtains the relation

$$2\mathrm{i}\ \sin\big(2\pi\rho\,(\lambda - \lambda^{-1})\big)\ \mathsf{e}^{2\pi\mathrm{i}\sigma' p_2}\ \boldsymbol{S}^{(2,3)}_{\sigma',+}(\lambda)\ \boldsymbol{S}^{(2,3)}_{\sigma',-}(\lambda)$$

$$= \boldsymbol{S}^{(1,2)}_{-,-\sigma'}(\lambda)\ \boldsymbol{S}^{(1,2)}_{+,-\sigma'}(\lambda \mathsf{e}^{-\mathrm{i}\pi\delta})\ \mathsf{e}^{-2\pi\mathrm{i}p_1} - \boldsymbol{S}^{(1,2)}_{+,-\sigma'}(\lambda)\ \boldsymbol{S}^{(1,2)}_{-,-\sigma'}(\lambda \mathsf{e}^{-\mathrm{i}\pi\delta})\ \mathsf{e}^{2\pi\mathrm{i}p_1}$$

which is equivalent to (6.99a) upon the substitution (6.98). One can repeat the above steps excluding $\boldsymbol{S}^{(2,3)}$ from (A.1). This leads to the relation (6.99c).

Finally, express $\boldsymbol{S}^{(1,2)}_{\sigma,\sigma'}(\lambda \mathsf{e}^{-\mathrm{i}\pi\delta})$ from (A.1) as

$$\boldsymbol{S}^{(1,2)}_{\sigma,\sigma'}(\lambda \mathsf{e}^{-\mathrm{i}\pi\delta}) = -\sum_{\sigma''=\pm} \mathsf{e}^{2\pi\mathrm{i}(\sigma p_1 + \sigma' p_2) + 2\pi\mathrm{i}\sigma''\rho\,(\lambda-\lambda^{-1})}\ \boldsymbol{S}^{(1,3)}_{\sigma,\sigma''}(\lambda)\ \boldsymbol{S}^{(3,2)}_{\sigma'',\sigma'}(\lambda)$$

and $\boldsymbol{S}^{(1,2)}_{\sigma,\sigma'}(\lambda)$ from (A.5) (with shifted spectral parameter)

$$\boldsymbol{S}^{(1,2)}_{\sigma,\sigma'}(\lambda) = \sum_{\sigma''=\pm} \boldsymbol{S}^{(1,3)}_{\sigma,\sigma''}(\lambda)\ \boldsymbol{S}^{(3,2)}_{\sigma'',\sigma'}(\lambda)\ .$$

Then the product $\boldsymbol{S}^{(1,3)}_{\sigma,-}\boldsymbol{S}^{(3,2)}_{-,\sigma'}$ can be excluded from the above equations, leading to

$$2\mathrm{i}\ \sin\big(2\pi\rho\,(\lambda - \lambda^{-1})\big)\ \boldsymbol{S}^{(1,3)}_{\sigma,\sigma''}(\lambda)\ \boldsymbol{S}^{(3,2)}_{\sigma'',\sigma'}(\lambda)$$

$$= \mathsf{e}^{-2\pi\mathrm{i}(\sigma p_1 + \sigma' p_2)}\ \boldsymbol{S}^{(1,2)}_{\sigma,\sigma'}(\lambda \mathsf{e}^{-\mathrm{i}\pi\delta}) + \mathsf{e}^{-2\pi\mathrm{i}\sigma''\rho\,(\lambda-\lambda^{-1})}\ \boldsymbol{S}^{(1,2)}_{\sigma,\sigma'}(\lambda)$$

which is equivalent to (6.99f) upon the substitution (6.98).

For conformal case to obtain functional relations for functions $\mathtt{W}^{(i)}$ we must use relations (6.67) instead of (6.98).

Bibliography

[1] L. B. Okun, *Leptons and Quarks.* Amsterdam, Netherlands: North-Holland, 1982.

[2] S. Weinberg, "The making of the Standard Model," *The European Physical Journal C - Particles and Fields*, vol. 34, no. 1, pp. 5–13, 2004.

[3] L. Faddeev and V. Popov, "Feynman diagrams for the Yang-Mills field," *Physics Letters B*, vol. 25, no. 1, pp. 29 – 30, 1967.

[4] B. S. DeWitt, "Theory of Radiative Corrections for Non-Abelian Gauge Fields," *Phys. Rev. Lett.*, vol. 12, pp. 742–746, 1964.

[5] B. S. DeWitt, "Quantum Theory of Gravity. II. The Manifestly Covariant Theory," *Phys. Rev.*, vol. 162, pp. 1195–1239, 1967.

[6] J. Goldstone, "Field theories with "superconductor" solutions," *Il Nuovo Cimento (1955-1965)*, vol. 19, no. 1, pp. 154–164, 1961.

[7] J. Goldstone, A. Salam, and S. Weinberg, "Broken symmetries," *Phys. Rev.*, vol. 127, pp. 965–970, 1962.

[8] P. Higgs, "Broken symmetries, massless particles and gauge fields," *Physics Letters*, vol. 12, no. 2, pp. 132 – 133, 1964.

[9] P. W. Higgs, "Broken symmetries and the masses of gauge bosons," *Phys. Rev. Lett.*, vol. 13, pp. 508–509, 1964.

[10] P. W. Higgs, "Spontaneous symmetry breakdown without massless bosons," *Phys. Rev.*, vol. 145, pp. 1156–1163, 1966.

[11] F. Englert and R. Brout, "Broken symmetry and the mass of gauge vector mesons," *Phys. Rev. Lett.*, vol. 13, pp. 321–323, 1964.

[12] S. Weinberg, "A model of leptons," *Phys. Rev. Lett.*, vol. 19, pp. 1264–1266, 1967.

[13] A. Salam and J. A. Strathdee, "On the zero energy behavior of Regge poles and residues," *Conf. Proc.*, vol. C680519, pp. 59–103, 1968.

[14] M. Gell-Mann, “A schematic model of baryons and mesons,” *Physics Letters*, vol. 8, no. 3, pp. 214 – 215, 1964.

[15] G. Zweig, “An SU_3 model for strong interaction symmetry and its breaking; Version 2.” CERN preprint, 1964.

[16] C. N. Yang and R. L. Mills, “Conservation of isotopic spin and isotopic gauge invariance,” *Phys. Rev.*, vol. 96, pp. 191–195, 1954.

[17] C. Patrignani *et al.*, “Review of Particle Physics,” *Chin. Phys.*, vol. C40, no. 10, p. 100001, 2016.

[18] C. W. Bauer, S. Fleming, D. Pirjol, and I. W. Stewart, “An effective field theory for collinear and soft gluons: Heavy to light decays,” *Phys. Rev. D*, vol. 63, p. 114020, 2001, `arXiv:hep-ph/0011336 [hep-ph]`.

[19] H. Georgi, “An effective field theory for heavy quarks at low energies,” *Physics Letters B*, vol. 240, no. 3, pp. 447 – 450, 1990.

[20] D. Diakonov, “Topology and confinement,” *Nuclear Physics B - Proceedings Supplements*, vol. 195, pp. 5 – 45, 2009, `arXiv:0906.2456 [hep-ph]`. Proceedings of the 47th Internationale Universitätswochen für Theoretische Physik.

[21] L. D. Faddeev, “Mass in Quantum Yang-Mills Theory,” 2009, `arXiv:0911.1013 [math-ph]`.

[22] G. ’t Hooft, “Renormalizable Lagrangians for Massive Yang-Mills Fields,” *Nucl. Phys.*, vol. B35, pp. 167–188, 1971.

[23] B. W. Lee and J. Zinn-Justin, “Spontaneously broken gauge symmetries. i. preliminaries,” *Phys. Rev. D*, vol. 5, pp. 3121–3137, 1972.

[24] G. ’t Hooft and M. Veltman, “Regularization and renormalization of gauge fields,” *Nuclear Physics B*, vol. 44, no. 1, pp. 189 – 213, 1972.

[25] G. ’t Hooft and M. J. G. Veltman, “Combinatorics of gauge fields,” *Nucl. Phys.*, vol. B50, pp. 318–353, 1972.

[26] K. G. Wilson, “The renormalization group and critical phenomena,” *Rev. Mod. Phys.*, vol. 55, pp. 583–600, 1983.

[27] K. G. Wilson and J. Kogut, “The renormalization group and the ϵ expansion,” *Physics Reports*, vol. 12, no. 2, pp. 75 – 199, 1974.

[28] M. Gell-Mann and F. E. Low, “Quantum electrodynamics at small distances,” *Phys. Rev.*, vol. 95, pp. 1300–1312, 1954.

[29] C. G. Callan, “Broken scale invariance in scalar field theory,” *Phys. Rev. D*, vol. 2, pp. 1541–1547, 1970.

[30] K. Symanzik, “Small distance behaviour in field theory and power counting,” *Comm. Math. Phys.*, vol. 18, no. 3, pp. 227–246, 1970.

[31] C. G. Callan, S. Coleman, and R. Jackiw, “A new improved energy-momentum tensor,” *Annals of Physics*, vol. 59, no. 1, pp. 42 – 73, 1970.

[32] D. J. Gross and F. Wilczek, “Ultraviolet behavior of non-abelian gauge theories,” *Phys. Rev. Lett.*, vol. 30, pp. 1343–1346, 1973.

[33] D. J. Gross and F. Wilczek, “Asymptotically free gauge theories. i,” *Phys. Rev. D*, vol. 8, pp. 3633–3652, 1973.

[34] H. D. Politzer, “Reliable perturbative results for strong interactions?,” *Phys. Rev. Lett.*, vol. 30, pp. 1346–1349, 1973.

[35] H. Fritzsch, M. Gell-Mann, and H. Leutwyler, “Advantages of the Color Octet Gluon Picture,” *Phys. Lett.*, vol. 47B, pp. 365–368, 1973.

[36] N. N. Bogolyubov and D. V. Shirkov, “INTRODUCTION TO THE THEORY OF QUANTIZED FIELDS,” *Intersci. Monogr. Phys. Astron.*, vol. 3, pp. 1–720, 1959.

[37] A. Polyakov, “Compact gauge fields and the infrared catastrophe,” *Physics Letters B*, vol. 59, no. 1, pp. 82 – 84, 1975.

[38] A. Belavin, A. Polyakov, A. Schwartz, and Y. Tyupkin, “Pseudoparticle solutions of the Yang-Mills equations,” *Physics Letters B*, vol. 59, no. 1, pp. 85 – 87, 1975.

[39] G. V. Dunne and M. Ünsal, “Resurgence and trans-series in quantum field theory: the $\mathbb{CP}^{N-1}$ model,” *Journal of High Energy Physics*, vol. 2012, no. 11, p. 170, 2012, `arXiv:1210.2423 [hep-th]`.

[40] N. A. Nekrasov, “Seiberg-Witten prepotential from instanton counting,” *Adv. Theor. Math. Phys.*, vol. 7, no. 5, pp. 831–864, 2003, `arXiv:hep-th/0206161`.

[41] V. Novikov, M. Shifman, A. Vainshtein, and V. Zakharov, “Two-dimensional sigma models: Modelling non-perturbative effects in quantum chromodynamics,” *Physics Reports*, vol. 116, no. 3, pp. 103 – 171, 1984.

[42] A. M. Polyakov, “Gauge Fields and Strings,” *Contemp. Concepts Phys.*, vol. 3, pp. 1–301, 1987.

[43] A. Polyakov and P. Wiegmann, "Theory of nonabelian goldstone bosons in two dimensions," *Physics Letters B*, vol. 131, no. 1, pp. 121 – 126, 1983.

[44] P. Wiegmann, "Exact solution of the $O(3)$ nonlinear σ-model," *Physics Letters B*, vol. 152, no. 3, pp. 209 – 214, 1985.

[45] P. Hasenfratz, M. Maggiore, and F. Niedermayer, "The exact mass gap of the $O(3)$ and $O(4)$ non-linear σ-models in $d = 2$," *Physics Letters B*, vol. 245, no. 3, pp. 522 – 528, 1990.

[46] P. Hasenfratz and F. Niedermayer, "The exact mass gap of the $O(N)$ σ-model for arbitrary $N \geq 3$ in $d = 2$," *Physics Letters B*, vol. 245, no. 3, pp. 529 – 532, 1990.

[47] V. A. Fateev, I. V. Frolov, and A. S. Shvarts, "Quantum Fluctuations of Instantons in the Nonlinear Sigma Model," *Nucl. Phys.*, vol. B154, pp. 1–20, 1979.

[48] A. Bukhvostov and L. Lipatov, "Instanton-anti-instanton interaction in the $O(3)$ non-linear σ model and an exactly soluble fermion theory," *Nuclear Physics B*, vol. 180, no. 1, pp. 116 – 140, 1981.

[49] H. Saleur, "The long delayed solution of the Bukhvostov-Lipatov model," *Journal of Physics A: Mathematical and General*, vol. 32, no. 18, p. L207, 1999, `arXiv:hep-th/9811023 [hep-th]`.

[50] R. J. Baxter, *Exactly solved models in statistical mechanics.* Academic Press, 1982.

[51] P. Dorey and R. Tateo, "Anharmonic oscillators, the thermodynamic Bethe ansatz and nonlinear integral equations," *Journal of Physics A: Mathematical and General*, vol. 32, no. 38, p. L419, 1999, `arXiv:hep-th/9812211 [hep-th]`.

[52] V. V. Bazhanov, S. L. Lukyanov, and A. B. Zamolodchikov, "Spectral Determinants for Schrödinger Equation and Q-Operators of Conformal Field Theory," *Journal of Statistical Physics*, vol. 102, no. 3, pp. 567–576, 2001, `arXiv:hep-th/9812247 [hep-th]`.

[53] L. D. Faddeev and L. A. Takhtajan, "HAMILTONIAN METHODS IN THE THEORY OF SOLITONS," *BERLIN, GERMANY: SPRINGER (1987) 592 P. (SPRINGER SERIES IN SOVIET MATHEMATICS)*, 1987.

[54] J. Suzuki, "Functional Relations in Stokes Multipliers—Fun with x**6+alpha*x**2 potential," *Journal of Statistical Physics*, vol. 102, no. 3, pp. 1029–1047, 2001, `arXiv:quant-ph/0003066 [quant-ph]`.

[55] V. V. Bazhanov, A. N. Hibberd, and S. Khoroshkin, "Integrable structure of W_3 Conformal Field Theory, Quantum Boussinesq Theory and Boundary Affine Toda Theory," *Nuclear Physics B*, vol. 622, no. 3, pp. 475 – 547, 2002, `arXiv:hep-th/0105177 [hep-th]`.

[56] V. V. Bazhanov, S. L. Lukyanov, and A. B. Zamolodchikov, "Higher level eigenvalues of Q operators and Schroedinger equation," *Adv. Theor. Math. Phys.*, vol. 7, no. 4, pp. 711–725, 2003, `arXiv:hep-th/0307108 [hep-th]`.

[57] D. Fioravanti, "Geometrical loci and CFTs via the Virasoro symmetry of the mKdV-SG hierarchy: an excursus," *Physics Letters B*, vol. 609, no. 1, pp. 173 – 179, 2005, `arXiv:hep-th/0408079 [hep-th]`.

[58] P. Dorey, C. Dunning, and R. Tateo, "The ODE/IM correspondence," *Journal of Physics A: Mathematical and Theoretical*, vol. 40, no. 32, p. R205, 2007, `arXiv:hep-th/0703066 [hep-th]`.

[59] P. Dorey, C. Dunning, D. Masoero, J. Suzuki, and R. Tateo, "Pseudo-differential equations, and the Bethe ansatz for the classical Lie algebras," *Nuclear Physics B*, vol. 772, no. 3, pp. 249 – 289, 2007, `arXiv:hep-th/0612298 [hep-th]`.

[60] B. Feigin and E. Frenkel, "Quantization of soliton systems and Langlands duality," 2007, `arXiv:0705.2486 [math.QA]`.

[61] S. L. Lukyanov and A. B. Zamolodchikov, "Quantum sine(h)-Gordon model and classical integrable equations," *Journal of High Energy Physics*, vol. 2010, no. 7, p. 8, 2010, `arXiv:1003:5333 [math-ph]`.

[62] P. Dorey, S. Faldella, S. Negro, and R. Tateo, "The Bethe ansatz and the Tzitzéica–Bullough–Dodd equation," *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, vol. 371, no. 1989, 2013, `arXiv:1209.5517 [math-ph]`.

[63] D. Masoero, A. Raimondo, and D. Valeri, "Bethe Ansatz and the Spectral Theory of Affine Lie Algebra-Valued Connections I. The simply-laced Case," *Communications in Mathematical Physics*, vol. 344, no. 3, pp. 719–750, 2016, `arXiv:1501.07421 [math-ph]`.

[64] K. Ito and C. Locke, "ODE/IM correspondence and modified affine Toda field equations," *Nuclear Physics B*, vol. 885, pp. 600 – 619, 2014, `arXiv:1502.00906 [hep-th]`.

[65] V. V. Bazhanov, S. L. Lukyanov, and B. A. Runov, "Vacuum energy of the Bukhvostov–Lipatov model," *Nuclear Physics B*, vol. 911, pp. 863 – 889, 2016, `arXiv:1607.04839 [hep-th]`.

[66] V. V. Bazhanov, S. L. Lukyanov, and B. A. Runov, "Bukhvostov–Lipatov model and quantum-classical duality," *Nuclear Physics B*, vol. 927, pp. 468 – 515, 2018, `arXiv:1711:09021 [hep-th]`.

[67] D. Volin, "From the mass gap at large N to the non-borel-summability in $O(3)$ and $O(4)$ sigma models," *Phys. Rev. D*, vol. 81, p. 105008, 2010, `arXiv:0904:2744 [hep-th]`.

[68] A. I. Vaĭnshteĭn, V. I. Zakharov, V. A. Novikov, and M. A. Shifman, "ABC of instantons," *Soviet Physics Uspekhi*, vol. 25, no. 4, p. 195, 1982.

[69] A. M. Polyakov and A. A. Belavin, "Metastable States of Two-Dimensional Isotropic Ferromagnets," *JETP Lett.*, vol. 22, pp. 245–248, 1975. [Pisma Zh. Eksp. Teor. Fiz.22,503(1975)].

[70] S. R. Coleman, "The Quantum Sine-Gordon Equation as the Massive Thirring Model," *Phys. Rev.*, vol. D11, p. 2088, 1975.

[71] J. Fröhlich, "Classical and quantum statistical mechanics in one and two dimensions: Two-component Yukawa — and Coulomb systems," *Communications in Mathematical Physics*, vol. 47, no. 3, pp. 233–268, 1976.

[72] S. Mandelstam, "Soliton Operators for the Quantized Sine-Gordon Equation," *Phys. Rev.*, vol. D11, p. 3026, 1975.

[73] V. Fateev, "The sigma model (dual) representation for a two-parameter family of integrable quantum field theories," *Nuclear Physics B*, vol. 473, no. 3, pp. 509 – 538, 1996.

[74] S. L. Lukyanov, "ODE/IM correspondence for the Fateev model," *Journal of High Energy Physics*, vol. 2013, no. 12, p. 12, 2013, `arXiv:1303.2566 [hep-th]`.

[75] S. L. Lukyanov, "The integrable harmonic map problem versus Ricci flow," *Nuclear Physics B*, vol. 865, no. 2, pp. 308 – 329, 2012, `arXiv:1205.3201 [hep-th]`.

[76] V. V. Bazhanov and S. L. Lukyanov, "From Fuchsian differential equations to integrable QFT," *Journal of Physics A: Mathematical and Theoretical*, vol. 47, no. 46, p. 462002, 2014, `arXiv:1310.8082 [math-ph]`.

[77] V. V. Bazhanov and S. L. Lukyanov, “Integrable structure of quantum field theory: classical flat connections versus quantum stationary states,” *Journal of High Energy Physics*, vol. 2014, no. 9, p. 147, 2014, `arXiv:1310.4390 [hep/th]`.

[78] V. V. Bazhanov, G. A. Kotousov, and S. L. Lukyanov, “Winding vacuum energies in a deformed $O(4)$ sigma model,” *Nuclear Physics B*, vol. 889, pp. 817 – 826, 2014, `arXiv:1409.0449 [hep-th]`.

[79] V. V. Bazhanov, G. A. Kotousov, and S. L. Lukyanov, “Quantum transfer-matrices for the sausage model,” *Journal of High Energy Physics*, vol. 2018, no. 1, p. 21, 2018, `arXiv:1706.09941 [hep-th]`.

[80] V. Fateev, E. Onofri, and A. Zamolodchikov, “Integrable deformations of the $O(3)$ sigma model. The sausage model,” *Nuclear Physics B*, vol. 406, no. 3, pp. 521 – 565, 1993.

[81] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal field theory*. Graduate texts in contemporary physics, New York, NY: Springer, 1997.

[82] V. S. Dotsenko and V. A. Fateev, “Conformal Algebra and Multipoint Correlation Functions in Two-Dimensional Statistical Models,” *Nucl. Phys.*, vol. B240, p. 312, 1984.

[83] A. Zamolodchikov, “Painlevé III and 2D polymers,” *Nuclear Physics B*, vol. 432, no. 3, pp. 427 – 456, 1994, `arXiv:hep-th/9409108`.

[84] A. Zamolodchikov, “Thermodynamic Bethe ansatz in relativistic models: Scaling 3-state Potts and Lee-Yang models,” *Nuclear Physics B*, vol. 342, no. 3, pp. 695 – 720, 1990.

[85] V. V. Bazhanov, S. L. Lukyanov, and A. B. Zamolodchikov, “Quantum field theories in finite volume: Excited state energies,” *Nuclear Physics B*, vol. 489, no. 3, pp. 487 – 531, 1997, `arXiv:hep-th/9607099 [hep-th]`.

[86] A. Voros, “Exact quantization condition for anharmonic oscillators (in one dimension),” *Journal of Physics A: Mathematical and General*, vol. 27, no. 13, p. 4653, 1994.

[87] V. V. Bazhanov, S. L. Lukyanov, and A. B. Zamolodchikov, “Integrable structure of conformal field theory, quantum KdV theory and Thermodynamic Bethe Ansatz,” *Communications in Mathematical Physics*, vol. 177, no. 2, pp. 381–398, 1996, `arXiv:hep-th/9412229 [hep-th]`.

[88] C. Destri and H. de Vega, “Unified approach to Thermodynamic Bethe Ansatz and finite size corrections for lattice models and field theories,” *Nuclear Physics B*, vol. 438, no. 3, pp. 413 – 454, 1995.

[89] C. Destri and H. J. de Vega, “New thermodynamic Bethe ansatz equations without strings,” *Phys. Rev. Lett.*, vol. 69, pp. 2313–2317, 1992.

[90] A. Klumper, M. T. Batchelor, and P. A. Pearce, “Central charges of the 6- and 19-vertex models with twisted boundary conditions,” *Journal of Physics A: Mathematical and General*, vol. 24, no. 13, p. 3111, 1991.

[91] K. Weierstrass, *Mathematische Werke*, vol. II. Berlin Mayer & Müller, 1985.

[92] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis.* Cambridge Mathematical Library, Cambridge University Press, 4 ed., 1996.

[93] M. D. Gould, J. R. Links, Y.-Z. Zhang, and I. Tsohantjis, “Twisted quantum affine superalgebra $U_q(sl(2|2)^{(2)})$, $U_q(osp(2|2))$ invariant R -matrices and a new integrable electronic model,” *Journal of Physics A: Mathematical and General*, vol. 30, no. 12, p. 4313, 1997.

[94] M. J. Martins and P. B. Ramos, “Solution of a supersymmetric model of correlated electrons,” *Phys. Rev. B*, vol. 56, pp. 6376–6379, 1997, `arXiv:hep-th/9704152 [hep-th]`.